Non-relativistic Schrödinger theory on q-deformed quantum spaces III

Scattering theory

Hartmut Wachter*

Max-Planck-Institute
for Mathematics in the Sciences
Inselstr. 22, D-04103 Leipzig

Arnold-Sommerfeld-Center
Ludwig-Maximilians-Universität
Theresienstr. 37, D-80333 München

Abstract

This is the third part of a paper about non-relativistic Schrödinger theory on q-deformed quantum spaces like the braided line or the three-dimensional q-deformed Euclidean space. Propagators for the free q-deformed particle are derived and their basic properties are discussed. A time-dependent formulation of scattering is proposed. In this respect, q-analogs of the Lippmann-Schwinger equation are given. Expressions for their iterative solutions are written down. It is shown how to calculate S-matrices and transition probabilities. Furthermore, attention is focused on the question what becomes of unitarity of S-matrices in a q-deformed setting. The examinations are concluded by a discussion of the interaction picture and its relation to scattering processes.

*e-mail: Hartmut.Wachter@physik.uni-muenchen.de

Contents

1 Introduction

It is an old idea to formulate quantum field theories on a space-time lattice, since it should lead to a natural cut-off in momentum space [1, 2]. In the literature one can find several attempts to attack this problem (see for example Refs. [3–8]). A more recent but very promising approach to discretize space-time is based on the theory of quantum groups and quantum spaces [9–39]. In our previous work we tried to develop the concepts of this theory in a way that allows their application to quantum theory [40–50].

This is the third and last part of an article which continues our former reasonings on physical aspects of the theory of quantum groups and quantum spaces. More concretely, the article is devoted to a non-relativistic Schrödinger theory on q-deformed quantum spaces as the braided line or the three-dimensional q-deformed Euclidean space. Before we summarize the content of its third part, let us briefly recall what we have already done in part I and II.

In part I we first presented the algebraic structure of the quantum spaces under consideration. Then we adapted our reasonings about q-deformed versions of analysis to the braided line and the three-dimensional q-deformed Euclidean space. Finally, we showed that this mathematical framework is compatible with basic concepts of quantum dynamics. Especially, we saw that the time-evolution operators for quantum spaces are of the same form as in the undeformed case. Furthermore, we gave q-analogs of the Schrödinger equation and the Heisenberg equations of motion.

Part II of the paper applies the reasonings of part I to the free non-relativistic particle. Especially, we considered q-analogs of momentum eigen-

functions and discussed their completeness, orthonormality, and dependence from time. In addition to this, we dealt with some more general aspects of quantum theory, i.e. the theorem of Ehrenfest and the conservation of probability. It was shown how these notions carry over to our q-deformed spaces.

Now, we come to the content of part III, which concludes our examinations of a q-deformed analog of non-relativistic Schrödinger theory. This part provides us with basic concepts of non-relativistic scattering theory on q-deformed quantum spaces. Towards this end, we reconsider the problem of time-evolution in quantum mechanics and treat it from the point of view of propagator theory. This task will be done in Sec. 2. In doing so, we obtain q-deformed expressions for the propagator of a free non-relativistic particle. Furthermore, we give a collection of the most important properties of these q-deformed propagators. After that we should be prepared to establish a q-deformed version of non-relativistic scattering theory. Section 3 covers this subject. More concretely, we first derive q-analogs of the Lippmann-Schwinger equation and solve them iteratively. This leads us to expansions that are relevant in perturbation theory. Furthermore, we introduce Green's functions for a particle interacting with a potential, write down their perturbation expansions, and discuss their basic properties. The matrix elements of these Green's functions can be used to calculate S-matrices and transition probabilities. Finally, we concern ourselves with the question what becomes of unitarity of S-matrices in a q-deformed setting. In Sec. 4 we revisit time-dependent scattering theory and treat it from the point of view provided by the interaction picture. Section 5 closes our considerations by a short conclusion.

The line of our reasonings is very similar to that in the undeformed case. However, there is one remarkable difference between a q-deformed theory and its undeformed limit, since in a q-deformed theory we have to distinguish different geometries. The reason for this lies in the fact that the braided tensor category in which the expressions of our theory live is not uniquely determined. This becomes more clear, if one realizes that each braided category is characterized by a so-called braiding mapping Ψ. The inverse Ψ^{-1} gives an equally good braiding, which leads to a second braided category being different from the first one. This observation is reflected in the occurrence of two differential calculi, different types of q-exponentials, q-integrals and so on. In this manner each braided category implies its own q-geometry, so we can write down different q-analogs of well-known physical laws.

Lastly, it should be noted that we assume the reader to be familiar with the results and conventions of part I and II. In this respect, it would

be helpful to have an idea of the reasonings in Sec. 3.1 or 3.2 of part I. Furthermore, we recommend to have a look at Sec. 2.2 and 2.3 of part II.

2 Propagators of the free particle

Once the wave function of a quantum system is known at a certain time the time-evolution operator enables us to find the wave function at any later time. However, there is another way to solve the time-evolution problem. It requires to know the so-called propagators. In this section we give expressions for the propagator of the q-deformed non-relativistic free particle and derive some of their basic properties.

We start from the expansions of wave functions in terms of plane waves. There are different q-geometries and each geometry leads to its own expansion. In part II of the article we found that

$$\begin{aligned}
(\phi_1)'_m(x^i) &= \frac{\kappa^n}{(\mathrm{vol}_1)^{1/2}} \int_{-\infty}^{+\infty} d_1^n p\, (c_1)'_{\kappa p} \overset{p|x}{\odot}_R (u_{\bar{R},L})_{\ominus_L p,m}(x^i), \\
(\phi_2)_m(x^i) &= \frac{\kappa^{-n}}{(\mathrm{vol}_2)^{1/2}} \int_{-\infty}^{+\infty} d_2^n p\, (\bar{u}_{R,\bar{L}})_{\ominus_R p,m}(x^i) \overset{x|p}{\odot}_L (c_2)_{\kappa^{-1}p},
\end{aligned} \tag{1}$$

$$\begin{aligned}
(\phi_1^*)_m(x^i) &= (\mathrm{vol}_1)^{1/2} \int_{-\infty}^{+\infty} d_1^n p\, (u_{\bar{R},L})_{p,m}(x^i) \overset{p}{\circledast} (c_1^*)_{\kappa^{-1}p}, \\
(\phi_2^*)'_m(x^i) &= (\mathrm{vol}_2)^{1/2} \int_{-\infty}^{+\infty} d_2^n p\, (c_2^*)'_{\kappa p} \overset{p}{\circledast} (\bar{u}_{R,\bar{L}})_{p,m}(x^i),
\end{aligned} \tag{2}$$

where the expansion coefficients can be calculated from the formulae

$$\begin{aligned}
(c_1)'_p &= (\mathrm{vol}_1)^{1/2} \int_{-\infty}^{+\infty} d_1^n x\, (\phi_1)'_m(x^i) \overset{x}{\circledast} (u_{\bar{R},L})_{p,m}(x^A, -q^{\zeta} t_x), \\
(c_2)_p &= (\mathrm{vol}_2)^{1/2} \int_{-\infty}^{+\infty} d_2^n x\, (\bar{u}_{R,\bar{L}})_{p,m}(x^A, -q^{-\zeta} t_x) \overset{x}{\circledast} (\phi_2)_m(x^i),
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
(c_1^*)_p &= \frac{1}{(\mathrm{vol}_1)^{1/2}} \int_{-\infty}^{+\infty} d_1^n x\, (u_{\bar{R},L})_{\ominus_{\bar{R}} p,m}(x^A, -q^{-\zeta} t_x) \overset{p|x}{\odot}_{\bar{L}} (\phi_1^*)_m(x^i), \\
(c_2^*)'_p &= \frac{1}{(\mathrm{vol}_2)^{1/2}} \int_{-\infty}^{+\infty} d_2^n x\, (\phi_2^*)'_m(x^i) \overset{x|p}{\odot}_{\bar{R}} (\bar{u}_{R,\bar{L}})_{\ominus_{\bar{L}} p,m}(x^A, -q^{\zeta} t_x).
\end{aligned} \tag{4}$$

Let us recall that for the quantum spaces under consideration the values

of κ and ζ are determined as follows:

(i) (braided line) $\kappa = q$, $\zeta = -1$,

(ii) (q-deformed Euclidean space in three dimensions) $\kappa = q^6$, $\zeta = 2$.

The q-analogs of plane-waves are related to q-deformed exponentials [43, 51, 52] by

$$
\begin{aligned}
(u_{\bar{R},L})_{p,m}(x^i) &= (\mathrm{vol}_1)^{-1/2} \exp(x^i|\mathrm{i}^{-1}p_j)_{\bar{R},L}\big|_{p_0=p^2/(2m)\overset{p}{\circledast}}, \\
(u_{R,\bar{L}})_{p,m}(x^i) &= (\mathrm{vol}_2)^{-1/2} \exp(x^i|\mathrm{i}^{-1}p_j)_{R,\bar{L}}\big|_{p_0=p^2/(2m)\overset{p}{\circledast}},
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
(\bar{u}_{\bar{R},L})_{p,m}(x^i) &= (\mathrm{vol}_1)^{-1/2} \exp(\mathrm{i}^{-1}p_j|x^i)_{\bar{R},L}\big|_{p_0=p^2/(2m)\overset{p}{\circledast}}, \\
(\bar{u}_{R,\bar{L}})_{p,m}(x^i) &= (\mathrm{vol}_2)^{-1/2} \exp(\mathrm{i}^{-1}p_j|x^i)_{R,\bar{L}}\big|_{p_0=p^2/(2m)\overset{p}{\circledast}}.
\end{aligned} \tag{6}
$$

The symbols $\odot_\gamma$, $\gamma \in \{L, \bar{L}, R, \bar{R}\}$, and $\circledast$ respectively denote braided products and star multiplication. Notice that braided products represent realizations of braiding mappings [45]. The symbol on top of a braided product indicates the tensor factors being involved in the braiding. It should also be mentioned that the volume elements are given by

$$
\begin{aligned}
\mathrm{vol}_1 &\equiv \int_{-\infty}^{+\infty} d_1^n x \int_{-\infty}^{+\infty} d_1^n p \exp(x^i|\mathrm{i}^{-1}p_j)_{\bar{R},L}\big|_{x^0=0} \\
&= \int_{-\infty}^{+\infty} d_1^n p \int_{-\infty}^{+\infty} d_1^n x \exp(\mathrm{i}^{-1}p_i|x^j)_{\bar{R},L}\big|_{x^0=0},
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\mathrm{vol}_2 &\equiv \int_{-\infty}^{+\infty} d_2^n x \int_{-\infty}^{+\infty} d_2^n p \exp(x^i|\mathrm{i}^{-1}p_j)_{R,\bar{L}}\big|_{x^0=0} \\
&= \int_{-\infty}^{+\infty} d_2^n p \int_{-\infty}^{+\infty} d_2^n x \exp(\mathrm{i}^{-1}p_i|x^j)_{R,\bar{L}}\big|_{x^0=0}.
\end{aligned} \tag{8}
$$

Expressions for calculating q-integrals over the whole position or momentum space were listed in part I of the paper [cf. Sec. 5 of part I].

As we already know from part I we have to distinguish different q-geometries. In this article, however, we restrict attention to certain geometries, only, since the expressions for the other geometries can be obtained from our results by applying the substitutions

$$
L \leftrightarrow \bar{L}, \quad R \leftrightarrow \bar{R}, \quad \kappa \leftrightarrow \kappa^{-1}, \quad q \leftrightarrow q^{-1},
$$

$$1\text{ (as label)} \leftrightarrow 2\text{ (as label)}, \quad \partial \leftrightarrow \hat{\partial}, \quad \triangleright \leftrightarrow \bar{\triangleright}, \quad \triangleleft \leftrightarrow \bar{\triangleleft}. \tag{9}$$

These correspondences are an example for so-called crossing symmetries, which are typical for q-deformation (see also Ref. [48]).

Now, we come to the derivation of free-particle propagators. Inserting the expressions for the expansion coefficients into the expansions of (1) and (2) leads us to integral operators that act on the initial wave functions to yield the final wave functions:

$$\begin{aligned}
(\phi_1)'_m(x^i) &= \int_{-\infty}^{+\infty} d_1^n y\, (\phi_1)'_m(y^j) \overset{y}{\circledast} (K_1)'_m(y^k, x^i), \\
(\phi_2)_m(x^i) &= \int_{-\infty}^{+\infty} d_2^n y\, (K_2)_m(x^i, y^j) \overset{y}{\circledast} (\phi_2)_m(y^k),
\end{aligned} \tag{10}$$

$$\begin{aligned}
(\phi_1^*)_m(x^i) &= \int_{-\infty}^{+\infty} d_1^n y\, (K_1^*)_m(x^i, y^j) \overset{y}{\circledast} (\phi_1^*)_m(\kappa y^A, t_y), \\
(\phi_2^*)'_m(x^i) &= \int_{-\infty}^{+\infty} d_2^n y\, (\phi_2^*)'_m(\kappa^{-1} y^A, t_y) \overset{y}{\circledast} (K_2^*)'_m(y^k, x^i).
\end{aligned} \tag{11}$$

The kernels of these integral operators are known as propagators and take the form

$$\begin{aligned}
(K_1)'_m(y^i, x^j) = \kappa^n \int_{-\infty}^{+\infty} & d_1^n p\, (u_{\bar{R},L})_{\kappa p,m}(y^A, -q^\zeta \kappa^{-2} t_y) \\
& \overset{p|x}{\odot}_R (u_{\bar{R},L})_{\ominus_L p,m}(x^B, t_x),
\end{aligned} \tag{12}$$

$$\begin{aligned}
(K_2)_m(x^i, y^j) = \kappa^{-n} \int_{-\infty}^{+\infty} & d_2^n p\, (\bar{u}_{R,\bar{L}})_{\ominus_R p,m}(x^A, t_x) \\
& \overset{x|p}{\odot}_L (\bar{u}_{R,\bar{L}})_{\kappa^{-1} p,m}(y^B, -q^{-\zeta} \kappa^2 t_y),
\end{aligned} \tag{13}$$

$$\begin{aligned}
(K_1^*)_m(x^i, y^j) = \kappa^n \int_{-\infty}^{+\infty} & d_1^n p\, (u_{\bar{R},L})_{p,m}(x^A, t_x) \\
& \overset{p|x}{\odot}_{\bar{L}} (u_{\bar{R},L})_{\ominus_{\bar{R}} p,m}(y^B, -q^{-\zeta} t_y),
\end{aligned} \tag{14}$$

$$\begin{aligned}
(K_2^*)'_m(y^i, x^j) = \kappa^{-n} \int_{-\infty}^{+\infty} & d_2^n p\, (\bar{u}_{R,\bar{L}})_{\ominus_R p,m}(y^A, -q^\zeta t_y) \\
& \overset{x|p}{\odot}_{\bar{R}} (\bar{u}_{R,\bar{L}})_{p,m}(x^B, t_x).
\end{aligned} \tag{15}$$

At this point it should be mentioned that we take the convention from part I and II that capital letters like A, B, etc. denote indices of space coordinates, only, i.e., for example, $x^i = (x^A, x^0) = (x^A, t)$.

We have to impose on the propagators the causality requirement, i.e. the wave function at time t can not be influenced by the wave functions at times $t' > t$. This leads us to the retarded Green's functions

$$\begin{aligned}(K_1)'_{m^+}(y^i, x^j) &\equiv \theta(t_x - t_y)(K_1)'_m(y^A, t_y; x^B, t_x),\\ (K_2)_{m^+}(x^i, y^j) &\equiv \theta(t_x - t_y)(K_2)_m(x^A, -t_x; y^B, -t_y),\end{aligned} \tag{16}$$

$$\begin{aligned}(K_1^*)_{m^+}(x^i, y^j) &\equiv \theta(t_x - t_y)(K_1^*)_m(x^A, t_x; y^B, t_y),\\ (K_2^*)'_{m^+}(y^i, x^j) &\equiv \theta(t_x - t_y)(K_2^*)'_m(y^A, -t_y; x^B, -t_x),\end{aligned} \tag{17}$$

where $\theta(t)$ stands for the Heaviside function

$$\theta(t) = \begin{cases} 1 & \text{if } t \geq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{18}$$

However, nothing prevents us from dealing with advanced Green's functions. They should be introduced in a way that it holds

$$\begin{aligned}(K_1)'_{m^-}(y^i, x^j) &\equiv (K_1)'_{m^+}(y^A, -t_y; x^B, -t_x),\\ (K_2)_{m^-}(x^i, y^j) &\equiv (K_2)_{m^+}(y^A, -t_y; x^B, -t_x),\end{aligned} \tag{19}$$

$$\begin{aligned}(K_1^*)_{m^-}(x^i, y^j) &\equiv (K_1^*)_{m^+}(y^A, -t_y; x^B, -t_x),\\ (K_2^*)'_{m^-}(y^i, x^j) &\equiv (K_2^*)'_{m^+}(y^A, -t_y; x^B, -t_x).\end{aligned} \tag{20}$$

Next, we wish to check that these Green's functions are solutions to Schrödinger equations with q-deformed delta functions as potential. This assertion follows from the same arguments as in the undeformed case:

$$\begin{aligned}\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (K_1)'_{m^+}(y^i, x^j) &= (\mathrm{i}\partial_0 \overset{t_x}{\triangleright} \theta(t_x - t_y))(K_1)'_m(y^i, x^j)\\ &\quad + \theta(t_x - t_y)(\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (K_1)'_m(y^i, x^j))\\ &= \mathrm{i}\delta(t_x - t_y)(K_1)'_m(y^i, x^j)\\ &\quad + \theta(t_x - t_y)(H_0 \overset{x}{\triangleright} (K_1)'_m(y^i, x^j))\\ &= \mathrm{i}\kappa^n (\mathrm{vol}_1)^{-1} \delta(t_x - t_y)\, \delta_1^n(\kappa y^A \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^B))\\ &\quad + H_0 \overset{x}{\triangleright} (K_1)'_{m^+}(y^i, x^j).\end{aligned} \tag{21}$$

The first step uses the definition of the retarded Green's function and the Leibniz rule for the time derivative. For the second step we make use of the fact that the time derivative of the Heaviside function is given by the classical delta function. Furthermore, we apply

$$\begin{aligned}
&\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (K_1)'_m(y^i, x^j) \\
&\quad = \kappa^n \int_{-\infty}^{+\infty} d_1^n p \, (u_{\bar{R},L})_{\kappa p,m}(y^A, -q^\zeta \kappa^{-2} t_y) \overset{p|x}{\odot}_R (\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (u_{\bar{R},L})_{\ominus_L p,m}(x^B, t_x)) \\
&\quad = \kappa^n \int_{-\infty}^{+\infty} d_1^n p \, (u_{\bar{R},L})_{\kappa p,m}(y^A, -q^\zeta \kappa^{-2} t_y) \overset{p|x}{\odot}_R (H_0 \overset{x}{\triangleright} (u_{\bar{R},L})_{\ominus_L p,m}(x^B, t_x)) \\
&\quad = H_0 \overset{x}{\triangleright} (K_1)'_m(y^i, x^j),
\end{aligned} \tag{22}$$

which is a direct consequence of the fact that q-deformed plane waves are solutions to Schrödinger equations (for the details see part II). For the last step in (21) we need

$$\begin{aligned}
&\lim_{t_x \to t_y} (K_1)'_m(y^i, x^j) \\
&\qquad = \kappa^n \int_{-\infty}^{+\infty} d_1^n p \, (u_{\bar{R},L})_{\kappa p,m}(y^A, -q^\zeta \kappa^{-2} t_x) \overset{p|x}{\odot}_R (u_{\bar{R},L})_{\ominus_L p,m}(x^B, t_x) \\
&\qquad = \kappa^n \int_{-\infty}^{+\infty} d_1^n p \, (u_{\bar{R},L})_{\kappa p,m}(y^A, 0) \overset{p|x}{\odot}_R (u_{\bar{R},L})_{\ominus_L p,m}(x^B, 0) \\
&\qquad = \kappa^n (\mathrm{vol}_1)^{-1} \, \delta_1^n(\kappa y^A \oplus_{\bar{R}} (\ominus_{\bar{R}} \, x^B)).
\end{aligned} \tag{23}$$

The second step in (23) uses that the time-dependent phase factors of the plane waves cancel out against each other. Finally, the last equality can be recognized as defining expression for a q-deformed delta function as it was given in Ref. [49].

In the case of the advanced Green's functions similar arguments lead us to

$$\begin{aligned}
\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (K_1)'_{m^-}(y^i, x^j) = &- \mathrm{i}\kappa^n (\mathrm{vol}_1)^{-1} \delta(t_y - t_x) \, \delta_1^n(\kappa y^A \oplus_{\bar{R}} (\ominus_{\bar{R}} \, x^B)) \\
&- H_0 \overset{x}{\triangleright} (K_1)'_{m^-}(y^i, x^j).
\end{aligned} \tag{24}$$

Repeating the same steps as above for the other geometries we additionally obtain

$$(K_2)_{m^\pm}(x^i, y^j) \overset{t_x}{\triangleleft} (\mathrm{i}\hat{\partial}_0)$$

$$
\begin{aligned}
&= \mp\ \mathrm{i}\kappa^{-n}(\mathrm{vol}_2)^{-1}\delta(\pm t_x \mp t_y)\, \delta_2^n((\ominus_{\bar{L}}\, x^A) \oplus_{\bar{L}} (\kappa^{-1} y^B)) \\
&\quad \mp\ (K_2)_{m^\pm}(x^i, y^j) \overset{x}{\triangleleft} H_0,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&-\ \mathrm{i}\partial_0 \overset{t_x}{\triangleright} (K_1^*)_{m^\pm}(x^i, y^j) \\
&\quad = \pm\ \mathrm{i}\kappa^{n}(\mathrm{vol}_1)^{-1}\delta(\pm t_x \mp t_y)\, \delta_1^n(x^A \oplus_{\bar{R}} (\ominus_{\bar{R}}\, y^B)) \\
&\quad \pm\ H_0 \overset{y}{\triangleright} (K_1^*)_{m^\pm}(x^i, y^j),
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&(K_2^*)'_{m^\pm}(y^i, x^j) \overset{t_x}{\triangleleft} (\mathrm{i}\hat{\partial}_0) \\
&\quad = \mp\ \mathrm{i}\kappa^{-n}(\mathrm{vol}_2)^{-1}\delta(\pm t_x \mp t_y)\, \delta_2^n((\ominus_{\bar{L}}\, y^A) \oplus_{\bar{L}} x^B) \\
&\quad \mp\ (K_2^*)_{m^\pm}(y^i, x^j) \overset{y}{\triangleleft} H_0.
\end{aligned} \tag{27}
$$

From the identities in (21) and (25) - (27) we can show that the Green's functions generate solutions to the inhomogeneous Schrödinger equations. In this manner, we have

$$
\begin{aligned}
\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (\psi_1)'_{\varrho^\pm}(x^i) \mp H_0 \overset{x}{\triangleright} (\psi_1)'_{\varrho^\pm}(x^i) &= \varrho^\pm(x^i), \\
\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (\psi_1^*)_{\varrho^\pm}(x^i) \mp H_0 \overset{x}{\triangleright} (\psi_1^*)_{\varrho^\pm}(x^i) &= \varrho^\pm(x^i),
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
(\psi_2)_{\varrho^\pm}(x^i) \overset{t_x}{\triangleleft} (\mathrm{i}\hat{\partial}_0) \pm (\psi_2)_{\varrho^\pm}(x^i) \overset{x}{\triangleleft} H_0 &= \varrho^\pm(x^i), \\
(\psi_2^*)'_{\varrho^\pm}(x^i) \overset{t_x}{\triangleleft} (\mathrm{i}\hat{\partial}_0) \pm (\psi_2^*)'_{\varrho^\pm}(x^i) \overset{x}{\triangleleft} H_0 &= \varrho^\pm(x^i),
\end{aligned} \tag{29}
$$

with

$$
\begin{aligned}
(\psi_1)'_{\varrho^\pm}(x^i) &= \mp\mathrm{i}\int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_1^n y\, \varrho^\pm(y^j) \overset{y}{\circledast} (K_1)'_{m^\pm}(y^k, x^i), \\
(\psi_2)_{\varrho^\pm}(x^i) &= \pm\mathrm{i}\int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_2^n y\, (K_2)_{m^\pm}(x^i, y^j) \overset{y}{\circledast} \varrho^\pm(y^k),
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
(\psi_1^*)_{\varrho^\pm}(x^i) &= \mp\mathrm{i}\int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_1^n y\, (K_1^*)_{m^\pm}(x^i, y^j) \overset{y}{\circledast} \varrho^\pm(\kappa y^A, t_y), \\
(\psi_2^*)'_{\varrho^\pm}(x^i) &= \pm\mathrm{i}\int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_2^n y\, \varrho^\pm(\kappa^{-1} y^A, t_y) \overset{y}{\circledast} (K_2^*)'_{m^\pm}(y^k, x^i).
\end{aligned} \tag{31}
$$

These assertions can be checked as follows:

$$
\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (\psi_1)'_{\varrho^\pm}(x^i) \mp H_0 \overset{x}{\triangleright} (\psi_1)'_{\varrho^\pm}(x^i)
$$

$$
\begin{aligned}
&= \mp\mathrm{i}\int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_1^n y\, \varrho^{\pm}(y^j) \overset{y}{\circledast} \big(\mathrm{i}\partial_0 \overset{t_x}{\triangleright} \mp H_0 \overset{x}{\triangleright}\big)(K_1)'_{m^{\pm}}(y^k, x^i) \\
&= \frac{\kappa^n}{\mathrm{vol}_1}\int_{-\infty}^{+\infty} dt_y\, \delta(\pm t_x \mp t_y) \int_{-\infty}^{+\infty} d_1^n y\, \varrho^{\pm}(y^j) \overset{y}{\circledast} \delta_1^n(\kappa y^A \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^B)) \\
&= \frac{1}{\mathrm{vol}_1}\int_{-\infty}^{+\infty} d_1^n y\, \varrho^{\pm}(\kappa^{-1} y^C, t_x) \overset{y}{\circledast} \delta_1^n(y^A \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^B)) \\
&= \varrho^{\pm}(x^i).
\end{aligned} \tag{32}
$$

For the first step we insert the expressions in (30) and (31). The second equality is a consequence of (21) and (24). The time integral vanishes due to the delta function and the last step makes use of the fundamental relations for q-deformed delta functions, which were derived in Ref. [50].

It is also worth recording here that the q-deformed Green's functions show a composition property. In analogy to their undeformed counterparts the q-deformed Green's functions satisfy

$$
\begin{aligned}
(K_1)'_{m^{\pm}}(y^i, x^j) &= \int_{-\infty}^{+\infty} d_1^n z\, (K_1)'_{m^{\pm}}(y^i, z^k) \overset{z}{\circledast} (K_1)'_{m^{\pm}}(z^l, x^j), \\
(K_2)_{m^{\pm}}(x^i, y^j) &= \int_{-\infty}^{+\infty} d_2^n z\, (K_2)_{m^{\pm}}(x^i, z^k) \overset{z}{\circledast} (K_2)_{m^{\pm}}(z^l, y^j),
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
(K_1^*)_{m^{\pm}}(x^i, y^j) &= \int_{-\infty}^{+\infty} d_1^n z\, (K_1^*)_{m^{\pm}}(x^i, z^k) \overset{z}{\circledast} (K_1^*)_{m^{\pm}}(\kappa z^A, t_z; y^j), \\
(K_2^*)'_{m^{\pm}}(y^i, x^j) &= \int_{-\infty}^{+\infty} d_2^n z\, (K_2^*)'_{m^{\pm}}(y^i; \kappa^{-1} z^A, t_z) \overset{z}{\circledast} (K_2^*)'_{m^{\pm}}(z^l, x^j).
\end{aligned} \tag{34}
$$

These identities can be proved in a rather straightforward manner. The following calculation shall serve as an example:

$$
\begin{aligned}
(\phi_1)'_m(x^i) &= \int_{-\infty}^{+\infty} d_1^n z\, (\phi_1)'_m(z^j) \overset{z}{\circledast} (K_1)'_{m^+}(z^k, x^i) \\
&= \int_{-\infty}^{+\infty} d_1^n z \int_{-\infty}^{+\infty} d_1^n y\, (\phi_1)'_m(y^k) \overset{y}{\circledast} (K_1)'_{m^+}(y^l, z^j) \overset{z}{\circledast} (K_1)'_{m^+}(z^k, x^i) \\
&= \int_{-\infty}^{+\infty} d_1^n y\, (\phi_1)'_m(y^k) \overset{y}{\circledast} \int_{-\infty}^{+\infty} d_1^n z\, (K_1)'_{m^+}(y^l, z^j) \overset{z}{\circledast} (K_1)'_{m^+}(z^k, x^i).
\end{aligned} \tag{35}
$$

Last but not least let us say a few words about the conjugation properties of the Green's functions for the free non-relativistic particle on q-deformed

quantum spaces. Concretely, we have ($i = 1, 2$)

$$\begin{aligned}\overline{(K_i)_{m^\pm}(x^k, y^l)} &= \theta(\pm t_x \mp t_y)\,(K_i)'_m(y^B, \mp t_y; x^A, \mp t_x) \equiv (\tilde{K}_i)'_{m^\pm}(x^k, y^l),\\ \overline{(K_i)'_{m^\pm}(y^k, x^l)} &= \theta(\pm t_x \mp t_y)\,(K_i)_m(x^B, \pm t_x; y^A, \pm t_y) \equiv (\tilde{K}_i)_{m^\pm}(y^k, x^l),\end{aligned} \tag{36}$$

$$\begin{aligned}\overline{(K_i^*)_{m^\pm}(x^k, y^l)} &= \theta(\pm t_x \mp t_y)\,(K_i^*)'_m(y^B, \pm t_y; x^A, \pm t_x) \equiv (\tilde{K}_i^*)'_{m^\pm}(x^k, y^l),\\ \overline{(K_i^*)'_{m^\pm}(y^k, x^l)} &= \theta(\pm t_x \mp t_y)\,(K_i^*)_m(x^B, \mp t_x; y^A, \mp t_y) \equiv (\tilde{K}_i^*)_{m^\pm}(y^k, x^l).\end{aligned} \tag{37}$$

To check these relations, one can proceed as follows:

$$\begin{aligned}&\overline{(K_1)'_{m^+}(y^i, x^j)} = \overline{\theta(t_x - t_y)(K_1)'_m(y^i, x^j)} = \theta(t_x - t_y)\overline{(K_1)'_m(y^i, x^j)}\\ &= \theta(t_x - t_y)\kappa^n \int_{-\infty}^{+\infty} d_1^n p\, \overline{(u_{\bar{R},L})_{\kappa p,m}(y^A, -q^\zeta \kappa^{-2} t_y) \overset{p|x}{\odot}_R (u_{\bar{R},L})_{\ominus_L p,m}(x^B, t_x)}\\ &= \theta(t_x - t_y)\kappa^n \int_{-\infty}^{+\infty} d_1^n p\, (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}} p,m}(x^B, t_x) \overset{p|x}{\odot}_{\bar{L}} (\bar{u}_{\bar{R},L})_{\kappa p,m}(y^A, -q^\zeta \kappa^{-2} t_y)\\ &= \theta(t_x - t_y)(K_1)_m(x^B, y^A).\end{aligned} \tag{38}$$

Notice that for the third and fourth step we took into account the conjugation properties of the elements of q-analysis, as they were given in Ref. [48].

In (36) and (37) we introduced propagators with a tilde. They belong to the instruction that particles with positive energy travel backwards in time, while those with negative energy move forward in time. In this sense, the new propagators do not conform with the usual agreement in physics that particles with negative energy travel backwards in time. However, this requirement is fulfilled by the propagators without a tilde. In our formalism we decided to assign $(\phi_i)'_m(t)$ and $(\phi_i^*)_m(t)$ a positive energy, whereas $(\phi_i^*)'_m(t)$ and $(\phi_i)_m(t)$ correspond to negative energies. Notice that in complete analogy to the undeformed case changing the sign of the time variable from plus to minus transforms wave functions with positive energy to those with negative energy and vice versa.

3 Elements of scattering theory

3.1 The Lippmann-Schwinger equations

In scattering theory one typically considers the situation that the time translation generator H can be divided into a free-particle Hamiltonian H_0 and an interaction V. However, in our formalism things become slightly more difficult, since we have to distinguish the following versions of time translation operators:

$$H = H_0 + V, \qquad H' = q^{-\zeta} H_0 + V(x^A), \qquad H'' = q^{\zeta} H_0 + V(x^A). \tag{39}$$

In this respect, it is our aim to seek solutions to the Schrödinger equations

$$\begin{aligned} \mathrm{i}\partial_0 \overset{t}{\triangleright} (\psi_1)'_{m^+}(x^i) &= H' \overset{x}{\triangleright} (\psi_1)'_{m^+}(x^i), \\ \mathrm{i}\partial_0 \overset{t}{\triangleright} (\psi_1^*)_{m^+}(x^i) &= H \overset{x}{\triangleright} (\psi_1^*)_{m^+}(x^i), \end{aligned} \tag{40}$$

$$\begin{aligned} \mathrm{i}\hat{\partial}_0 \overset{t}{\bar{\triangleright}} (\psi_2)'_{m^+}(x^i) &= H'' \overset{x}{\bar{\triangleright}} (\psi_2)'_{m^+}(x^i), \\ \mathrm{i}\hat{\partial}_0 \overset{t}{\bar{\triangleright}} (\psi_2^*)_{m^+}(x^i) &= H \overset{x}{\bar{\triangleright}} (\psi_2^*)_{m^+}(x^i), \end{aligned} \tag{41}$$

and

$$\begin{aligned} (\psi_1)_{m^-}(x^i) \overset{t}{\bar{\triangleleft}} (\mathrm{i}\partial_0) &= (\psi_1)_{m^-}(x^i) \overset{x}{\bar{\triangleleft}} H', \\ (\psi_1^*)'_{m^-}(x^i) \overset{t}{\bar{\triangleleft}} (\mathrm{i}\partial_0) &= (\psi_1^*)'_{m^-}(x^i) \overset{x}{\bar{\triangleleft}} H, \end{aligned} \tag{42}$$

$$\begin{aligned} (\psi_2)_{m^-}(x^i) \overset{t}{\triangleleft} (\mathrm{i}\hat{\partial}_0) &= (\psi_2)_{m^-}(x^i) \overset{x}{\triangleleft} H'', \\ (\psi_2^*)'_{m^-}(x^i) \overset{t}{\triangleleft} (\mathrm{i}\hat{\partial}_0) &= (\psi_2^*)'_{m^-}(x^i) \overset{x}{\triangleleft} H, \end{aligned} \tag{43}$$

where we require for these solutions to be subject to the boundary conditions ($i = 1, 2$)

$$\begin{aligned} \lim_{t_x \to +\infty} ((\psi_i)_m(x^A, t_x) - (\phi_i)_m(x^A, t_x)) &= 0, \\ \lim_{t_x \to -\infty} ((\psi_i)'_m(x^A, t_x) - (\phi_i)'_m(x^A, t_x)) &= 0, \end{aligned} \tag{44}$$

$$\begin{aligned} \lim_{t_x \to -\infty} ((\psi_i^*)_m(x^A, t_x) - (\phi_i^*)_m(x^A, t_x)) &= 0, \\ \lim_{t_x \to +\infty} ((\psi_i^*)'_m(x^A, t_x) - (\phi_i^*)'_m(x^A, t_x)) &= 0. \end{aligned} \tag{45}$$

Essentially for us is the fact that the Schrödinger equations in (40)-(43) can be seen as free-particle Schrödinger equations with an inhomogeneous contribution. In this manner we have, for example,

$$\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (\psi_1)'_m(x^i) - q^{-\zeta} H_0 \overset{x}{\triangleright} (\psi_1)'_m(x^i) = \varrho(x^i), \tag{46}$$

with

$$\varrho(x^i) = V(x^j) \overset{x}{\circledast} (\psi_1)'_m(x^i) = (\psi_1)'_m(x^i) \overset{x}{\circledast} V(x^j). \tag{47}$$

Recalling the identities in (30) and (31) we find that a solution to (46) has to fulfill the so-called Lippmann-Schwinger equation

$$\begin{aligned}(\psi_1)'_{m^+}(x^i) = {} & (\phi_1)'_m(x^A, q^{-\zeta}t_x) \\ & - \mathrm{i} \int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_1^n y \, (\psi_1)'_{m^+}(y^j) \overset{y}{\circledast} V(y^k) \overset{y}{\circledast} (K_1)'_{m^+}(y^l; x^A, q^{-\zeta}t_x).\end{aligned} \tag{48}$$

Notice that each solution to this integral equation shows the correct boundary condition, since the Green's function vanishes as $t_x \to -\infty$. Repeating the above arguments for the other geometries we also find

$$\begin{aligned}(\psi_2)_{m^-}(x^i) = {} & (\phi_2)_m(x^A, q^{\zeta}t_x) \\ & - \mathrm{i} \int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_2^n y \, (K_2)_{m^-}(x^A, q^{\zeta}t_x; y^j) \\ & \qquad \overset{y}{\circledast} V(y^k) \overset{y}{\circledast} (\psi_2)_{m^-}(y^l),\end{aligned} \tag{49}$$

$$\begin{aligned}(\psi_1^*)_{m^+}(x^i) = {} & (\phi_1^*)_m(x^i) \\ & - \mathrm{i} \int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_1^n y \, (K_1^*)_{m^+}(x^i, y^j) \overset{y}{\circledast} V(\kappa y^A, t_y) \\ & \qquad \overset{y}{\circledast} (\psi_1^*)_{m^+}(\kappa y^B, t_y),\end{aligned} \tag{50}$$

$$\begin{aligned}(\psi_2^*)'_{m^-}(x^i) = {} & (\phi_2^*)'_m(x^i) \\ & - \mathrm{i} \int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_2^n y \, (\psi_2^*)'_{m^-}(\kappa^{-1} y^A, t_y) \overset{y}{\circledast} V(\kappa^{-1} y^B, t_y) \\ & \qquad \overset{y}{\circledast} (K_2^*)'_{m^-}(y^k, x^i).\end{aligned} \tag{51}$$

To solve the Lippmann-Schwinger equations it is sometimes convenient

to introduce new Green's functions, for which we require to hold

$$(\psi_1)'_{m^+}(x^i) = \lim_{t_y \to -\infty} \int_{-\infty}^{+\infty} d_1^n y \, (\phi_1)'_m(y^A, q^{-\zeta} t_y) \overset{y}{\circledast} (G_1)'_{m^+}(y^B, t_y; x^i),$$
$$(\psi_2)_{m^-}(x^i) = \lim_{t_y \to +\infty} \int_{-\infty}^{+\infty} d_2^n y \, (G_2)_{m^-}(x^i; y^A, t_y) \overset{y}{\circledast} (\phi_2)_m(y^B, q^{\zeta} t_y), \quad (52)$$

$$(\psi_1^*)_{m^+}(x^i) = \lim_{t_y \to -\infty} \int_{-\infty}^{+\infty} d_1^n y \, (G_1^*)_{m^+}(x^i; y^A, t_y) \overset{y}{\circledast} (\phi_1^*)_m(\kappa y^B, t_y),$$
$$(\psi_2^*)'_{m^-}(x^i) = \lim_{t_y \to +\infty} \int_{-\infty}^{+\infty} d_2^n y \, (\phi_2^*)'_m(\kappa^{-1} y^A, t_y) \overset{y}{\circledast} (G_2^*)'_{m^-}(y^B, t_y; x^i). \quad (53)$$

With these relations at hand we can rewrite the Lippmann-Schwinger equations in a way that enables us to read off equations for the new Green's functions:

$$\begin{aligned}
(\psi_1)'_{m^+}(x^i) = {}& (\phi_1)'_m(x^A, q^{-\zeta} t_x) \\
& - \mathrm{i} \int_{-\infty}^{+\infty} dt_y \int_{-\infty}^{+\infty} d_1^n y \, (\psi_1)'_{m^+}(y^j) \overset{y}{\circledast} V(y^k) \overset{y}{\circledast} (K_1)'_{m^+}(y^l; x^A, q^{-\zeta} t_x) \\
= {}& \lim_{t_{y_1} \to -\infty} \int_{-\infty}^{+\infty} d_1^n y_1 \, (\phi_1)'_m(y_1^A, t_y) \overset{y_1}{\circledast} (K_1)'_{m^+}(y_1^B, t_{y_1}; x^C, q^{-\zeta} t_x) \\
& - \lim_{t_{y_1} \to -\infty} \mathrm{i} \int_{-\infty}^{+\infty} dt_{y_2} \int_{-\infty}^{+\infty} d_1^n y_2 \int_{-\infty}^{+\infty} d_1^n y_1 \, (\phi_1)'_m(y_1^A, t_{y_1}) \\
& \overset{y_1}{\circledast} (G_1)'_{m^+}(y_1^B, t_{y_1}; y_2^j) \overset{y_2}{\circledast} V(y_2^k) \overset{y_2}{\circledast} (K_1)'_{m^+}(y_2^l; x^C, q^{-\zeta} t_x).
\end{aligned} \quad (54)$$

Comparing this result with the first relation in (52) finally yields

$$\begin{aligned}
(G_1)'_{m^+}(y^i, x^j) = {}& (K_1)'_{m^+}(y^i; x^A, q^{-\zeta} t_x) \\
& - \mathrm{i} \int_{-\infty}^{+\infty} dt_z \int_{-\infty}^{+\infty} d_1^n z \, (G_1)'_{m^+}(y^i, z^k) \overset{z}{\circledast} V(z^l) \overset{z}{\circledast} (K_1)'_{m^+}(z^r; x^A, q^{-\zeta} t_x).
\end{aligned} \quad (55)$$

Applying similar arguments to the other geometries leads us to

$$(G_2)_{m^-}(x^i, y^j) = (K_2)_{m^-}(x^A, q^{\zeta} t_x; y^j)$$

$$
\begin{aligned}
&- \mathrm{i}\int_{-\infty}^{+\infty} dt_z \int_{-\infty}^{+\infty} d_2^n z\, (K_2)_{m^-}(x^A, q^\zeta t_x;, z^k) \overset{z}{\circledast} V(z^l) \\
&\qquad\qquad \overset{z}{\circledast} (G_2)_{m^-}(z^r, y^j),
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
(G_1^*)_{m^+}(x^i, y^j) = {} & (K_1^*)_{m^+}(x^i, y^j) \\
&- \mathrm{i}\int_{-\infty}^{+\infty} dt_z \int_{-\infty}^{+\infty} d_1^n z\, (K_1^*)_{m^+}(x^i, z^k) \overset{z}{\circledast} V(\kappa z^A, t_z) \\
&\qquad\qquad \overset{z}{\circledast} (G_1^*)_{m^+}(\kappa z^B, t_z; y^j),
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
(G_2^*)'_{m^-}(y^i, x^j) = {} & (K_2^*)'_{m^-}(y^i, x^j) \\
&- \mathrm{i}\int_{-\infty}^{+\infty} dt_z \int_{-\infty}^{+\infty} d_2^n z\, (G_2^*)'_{m^-}(y^i, \kappa^{-1} z^A, t_z) \overset{z}{\circledast} V(\kappa^{-1} z^B, t_z) \\
&\qquad\qquad \overset{z}{\circledast} (K_2^*)'_{m^-}(z^k, x^j).
\end{aligned} \tag{58}
$$

Next, we would like to mention that the Green's functions satisfy

$$
\begin{aligned}
&\mathrm{i}\partial_0 \overset{t_x}{\triangleright} (G_1)'_{m^+}(y^i, x^j) - H' \overset{x}{\triangleright} (G_1)'_{m^+}(y^i, x^j) = \\
&\qquad = \mathrm{i} q^{-\zeta} \kappa^n (\mathrm{vol}_1)^{-1} \delta(q^{-\zeta} t_x - t_y)\, \delta_1^n(\kappa y^A \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^B)),
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
&(G_2)_{m^-}(x^i, y^j) \overset{t_x}{\triangleleft} (\mathrm{i}\hat{\partial}_0) - (G_2)_{m^-}(x^i, y^j) \overset{x}{\triangleleft} H'' = \\
&\qquad = \mathrm{i} q^{\zeta} \kappa^{-n} (\mathrm{vol}_2)^{-1} \delta(t_y - q^{\zeta} t_x)\, \delta_2^n((\ominus_{\bar{L}}\, x^A) \oplus_{\bar{L}} (\kappa^{-1} y^B)),
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
&\mathrm{i}\partial_0 \overset{t_y}{\triangleright} (G_1^*)_{m^+}(x^i, y^j) - H \overset{y}{\triangleright} (G_1^*)_{m^+}(x^i, y^j) = \\
&\qquad = \mathrm{i} \kappa^n (\mathrm{vol}_1)^{-1} \delta(t_x - t_y)\, \delta_1^n(x^A \oplus_{\bar{R}} (\ominus_{\bar{R}}\, y^B)),
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
&(G_2^*)'_{m^-}(y^i, x^j) \overset{t_y}{\triangleleft} (\mathrm{i}\hat{\partial}_0) - (G_2^*)'_{m^-}(y^i, x^j) \overset{y}{\triangleleft} H = \\
&\qquad = \mathrm{i} \kappa^{-n} (\mathrm{vol}_2)^{-1} \delta(t_y - t_x)\, \delta_2^n((\ominus_{\bar{L}}\, y^A) \oplus_{\bar{L}} x^B).
\end{aligned} \tag{62}
$$

To prove these identities we first substitute the expressions in (55)-(58) for the Green's functions and then apply the relations in (21) and (25)-(27).

It is also worth recording here that the Green's functions defined by the relations in (52) and (53) can alternatively be introduced by

$$
\begin{aligned}
&q^{-\zeta} \theta(q^{-\zeta} t_x - t_y) (\psi_1)'_{m^+}(x^i) = \\
&\qquad\quad = \int_{-\infty}^{+\infty} d_1^n y\, (\psi_1)'_{m^+}(y^A, t_y) \overset{y}{\circledast} (G_{m^+})_1(y^B, t_y; x^i),
\end{aligned} \tag{63}
$$

$$q^{\zeta}\theta(t_y - q^{\zeta}t_x)(\psi_2)_{m^-}(x^i) = \\ = \int_{-\infty}^{+\infty} d_2^n y\,(G_{m^-})_2(x^i; y^A, t_y) \overset{y}{\circledast} (\psi_2)_{m^-}(y^B, t_y), \tag{64}$$

$$\theta(t_x - t_y)(\psi_1^*)_{m^+}(x^i) = \\ = \int_{-\infty}^{+\infty} d_1^n y\,(G^*_{m^+})_1(x^i; y^A, t_y) \overset{y}{\circledast} (\psi_1^*)_{m^+}(\kappa y^B, t_y), \tag{65}$$

$$\theta(t_y - t_x)(\psi_2^*)'_{m^-}(x^i) = \\ = \int_{-\infty}^{+\infty} d_2^n y\,(\psi_2^*)'_{m^-}(\kappa^{-1} y^A, t_y) \overset{y}{\circledast} (G^*_{m^-})'_2(y^B, t_y; x^i). \tag{66}$$

To show that these definitions are indeed equivalent to those in (52) and (53) we apply Schrödinger operators to both sides of each equality in (63) and (66). This way, we obtain, for example,

$$\begin{aligned} &\big(\mathrm{i}\partial_0 \overset{t_x}{\triangleright} - H' \overset{x}{\triangleright}\big)\big(\theta(q^{-\zeta}t_x - t_y)(\psi_1)'_{m^+}(x^i)\big) = \\ &\quad = \mathrm{i}q^{-\zeta}\delta(q^{-\zeta}t_x - t_y)(\psi_1)'_{m^+}(x^i) + \theta(t_x - t_y)\big(\mathrm{i}\partial_0 \overset{t_x}{\triangleright} - H \overset{x}{\triangleright}\big)(\psi_1)'_{m^+}(x^i) \\ &\quad = \mathrm{i}q^{-\zeta}\delta(q^{-\zeta}t_x - t_y)(\psi_1)'_{m^+}(x^A, t_x). \end{aligned} \tag{67}$$

Due to the characteristic property of the Green's function [cf. Eq. (59)] this is equal to the expression

$$\begin{aligned} &\big(\mathrm{i}\partial_0 \overset{t_x}{\triangleright} - H' \overset{x}{\triangleright}\big) \int_{-\infty}^{+\infty} d_1^n y\,(\psi_1)'_{m^+}(y^A, t_y) \overset{y}{\circledast} (G_1)'_{m^+}(y^B, t_y; x^i) = \\ &= \int_{-\infty}^{+\infty} d_1^n y\,(\psi_1)'_m(y^A, t_y) \overset{y}{\circledast} \big(\mathrm{i}\partial_0 \overset{t_x}{\triangleright} - H' \overset{x}{\triangleright}\big)(G_1)'_{m^+}(y^B, t_y; x^i) \\ &= \mathrm{i}\delta(q^{-\zeta}t_x - t_y)\,\frac{q^{-\zeta}\kappa^n}{\mathrm{vol}_1} \int_{-\infty}^{+\infty} d_1^n y\,(\psi_1)'_{m^+}(y^A, t_y) \overset{y}{\circledast} \delta_1^n(\kappa y^B \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^C)) \\ &= \mathrm{i}q^{-\zeta}\delta(q^{-\zeta}t_x - t_y)\,(\psi_1)'_{m^+}(x^A, t_x). \end{aligned} \tag{68}$$

The Green's functions defined by the relations in (52) and (53) again show the composition property. Concretely, it holds

$$(G_1)'_{m^+}(y^A, t_y; x^B, t_x) = \\ = \int_{-\infty}^{+\infty} d_1^n z\,(G_1)'_{m^+}(y^A, t_y; z^C, t_z) \overset{z}{\circledast} (G_1)'_{m^+}(z^D, t_z; x^B, t_x), \tag{69}$$

$$(G_2)_{m^-}(y^A, t_y; x^B, t_x) = \\ = \int_{-\infty}^{+\infty} d_2^n z\, (G_2)_{m^-}(y^A, t_y; z^C, t_z) \overset{z}{\circledast} (G_2)_{m^-}(z^D, t_z; x^B, t_x), \tag{70}$$

and

$$(G_1^*)_{m^+}(x^A, t_x; y^B, t_y) = \\ = \int_{-\infty}^{+\infty} d_1^n z\, (G_1^*)_{m^+}(x^A, t_x; z^C, t_z) \overset{z}{\circledast} (G_1^*)_{m^+}(\kappa z^D, t_z; y^B, t_y), \tag{71}$$

$$(G_2^*)'_{m^-}(x^A, t_x; y^B, t_y) = \\ = \int_{-\infty}^{+\infty} d_2^n z\, (G_2^*)'_{m^-}(x^A, t_x; \kappa^{-1} z^C, t_z) \overset{z}{\circledast} (G_2^*)'_{m^-}(z^D, t_z; y^B, t_y), \tag{72}$$

where $t_x > t_z > t_y$. In addition to this, we have the identities

$$\int_{-\infty}^{+\infty} d_1^n z\, (\tilde{G}_1)'_{m^-}(y^A, t; z^C, t_-) \overset{z}{\circledast} (G_1)'_{m^+}(z^D, t_-; x^B, t) = \\ = \int_{-\infty}^{+\infty} d_1^n z\, (G_1)'_{m^+}(y^A, t; z^C, t_+) \overset{z}{\circledast} (\tilde{G}_1)'_{m^-}(z^D, t_+; x^B, t) \\ = \kappa^n (\mathrm{vol}_1)^{-1}\, \delta_1^n(\kappa y^A \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^B)), \tag{73}$$

$$\int_{-\infty}^{+\infty} d_2^n z\, (\tilde{G}_1)_{m^+}(y^A, t; z^C, t_-) \overset{z}{\circledast} (G_2)_{m^-}(z^D, t_-; x^B, t) = \\ = \int_{-\infty}^{+\infty} d_2^n z\, (G_2)_{m^-}(y^A, t; z^C, t_+) \overset{z}{\circledast} (\tilde{G}_2)_{m^+}(z^D, t_+; x^B, t) \\ = \kappa^{-n} (\mathrm{vol}_2)^{-1}\, \delta_2^n((\ominus_{\bar{L}}\, y^A) \oplus_{\bar{L}} (\kappa^{-1} x^B)), \tag{74}$$

and

$$\int_{-\infty}^{+\infty} d_1^n z\, (\tilde{G}_1^*)_{m^-}(x^A, t; z^C, t_+) \overset{z}{\circledast} (G_1^*)_{m^+}(\kappa z^D, t_+; y^B, t) = \\ = \int_{-\infty}^{+\infty} d_1^n z\, (G_1^*)_{m^+}(x^A, t; z^C, t_-) \overset{z}{\circledast} (\tilde{G}_1^*)_{m^-}(\kappa z^D, t_-; y^B, t) \\ = \kappa^n (\mathrm{vol}_1)^{-1}\, \delta_1^n(x^A \oplus_{\bar{R}} (\ominus_{\bar{R}}\, y^B)), \tag{75}$$

$$\int_{-\infty}^{+\infty} d_2^n z\, (\tilde{G}_2^*)'_{m^+}(x^A, t; \kappa^{-1} z^C, t_+) \overset{z}{\circledast} (G_2^*)'_{m^-}(z^D, t_+; y^B, t) =$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} d_2^n z\,(G_2^*)'_{m^-}(x^A, t; \kappa^{-1} z^C, t_-) \overset{z}{\circledast} (\tilde{G}_2^*)'_{m^+}(z^D, t_-; y^B, t) \\
&= \kappa^{-n} (\mathrm{vol}_2)^{-1}\, \delta_2^n((\ominus_{\bar{L}}\, x^A) \oplus_{\bar{L}}\, y^B), \qquad (76)
\end{aligned}
$$

where we now assume $t_+ > t > t_-$.

Notice that the expressions for the Green's functions with a tilde are obtained from those in (55)-(58) by replacing the free-particle Green's functions with the corresponding ones in (36) and (37):

$$
\begin{aligned}
(\tilde{G}_1)'_{m^-}(y^i, x^j) = {} & (\tilde{K}_1)'_{m^-}(y^i; x^A, q^{-\zeta} t_x) \\
& - \mathrm{i} \int_{-\infty}^{+\infty} dt_z \int_{-\infty}^{+\infty} d_1^n z\,(\tilde{G}_1)'_{m^+}(y^i, z^k) \overset{z}{\circledast} V(z^l) \\
& \qquad \overset{z}{\circledast} (\tilde{K}_1)'_{m^-}(z^r; x^A, q^{-\zeta} t_x), \qquad (77)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{G}_2)_{m^+}(x^i, y^j) = {} & (\tilde{K}_2)_{m^+}(x^A, q^{\zeta} t_x; y^j) \\
& - \mathrm{i} \int_{-\infty}^{+\infty} dt_z \int_{-\infty}^{+\infty} d_2^n z\,(\tilde{K}_2)_{m^+}(x^A, q^{\zeta} t_x; z^k) \overset{z}{\circledast} V(z^l) \\
& \qquad \overset{z}{\circledast} (\tilde{G}_2)_{m^+}(z^r, y^j), \qquad (78)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{G}_1^*)_{m^-}(x^i, y^j) = {} & (\tilde{K}_1^*)_{m^-}(x^i, y^j) \\
& - \mathrm{i} \int_{-\infty}^{+\infty} dt_z \int_{-\infty}^{+\infty} d_1^n z\,(\tilde{K}_1^*)_{m^-}(x^i, z^k) \overset{z}{\circledast} V(\kappa z^A, t_z) \\
& \qquad \overset{z}{\circledast} (\tilde{G}_1^*)_{m^-}(\kappa z^B, t_z; y^j), \qquad (79)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{G}_2^*)'_{m^+}(y^i, x^j) = {} & (\tilde{K}_2^*)'_{m^+}(y^i, x^j) \\
& - \mathrm{i} \int_{-\infty}^{+\infty} dt_z \int_{-\infty}^{+\infty} d_2^n z\,(\tilde{G}_2^*)'_{m^+}(y^i, \kappa^{-1} z^A, t_z) \overset{z}{\circledast} V(\kappa^{-1} z^B, t_z) \\
& \qquad \overset{z}{\circledast} (\tilde{K}_2^*)'_{m^+}(z^k, x^j). \qquad (80)
\end{aligned}
$$

The relations in (69)-(72) follow from the same reasonings as those in (33)-(34) if we take into account the identities in (64) and (66) . To prove the relations in (73)-(76), we can proceed as follows:

$$
\begin{aligned}
\psi(x^i) &= \int_{-\infty}^{+\infty} d_1^n y\,\psi(y^C, t_+) \overset{y}{\circledast} (\tilde{G}_1)'_{m^-}(y^B, t_+; x^A, t_x) \\
&= \int_{-\infty}^{+\infty} d_1^n z \int_{-\infty}^{+\infty} d_1^n y\,\psi(z^E, t) \overset{z}{\circledast} (G_1)'_{m^+}(z^D, t_x; y^C, t_+)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{y}{\circledast}(\tilde{G}_1)'_{m^-}(y^B, t_+; x^A, t_x) \\
= \int_{-\infty}^{+\infty} d_1^n z\, \psi(z^E, t_x) \overset{z}{\circledast} \int_{-\infty}^{+\infty} d_1^n y\, &(G_1)'_{m^+}(z^D, t_x; y^C, t_+) \\
&\overset{y}{\circledast}(\tilde{G}_1)'_{m^-}(y^B, t_+; x^A, t_x). \qquad (81)
\end{aligned}
$$

On the other hand, we have (see for example Ref. [49])

$$
\psi(x^i) = \frac{\kappa^n}{\mathrm{vol}_1} \int_{-\infty}^{+\infty} d_1^n z\, \psi(z^E, t_x) \overset{z}{\circledast} \delta_1^n(\kappa z^D \oplus_{\bar{R}} (\ominus_{\bar{R}}\, x^A)), \qquad (82)
$$

which shows us the validity of the first relation in (74).

Let us return to the Lippmann-Schwinger equations, once again. If the interaction V is small, we can solve them iteratively. In doing so we obtain q-analogs of the famous Born series:

$$
\begin{aligned}
(\psi_1)'_{m^+}(x^i) = {} & (\phi_1)'_m(x^A, q^{-\zeta} t_x) \\
& + \mathrm{i}^{-1} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_1^n y_1\, (\phi_1)'_m(y_1^A, q^{-\zeta} t_1) \overset{y_1}{\circledast} V(y_1^j) \\
& \qquad \overset{y_1}{\circledast} (K_1)'_{m^+}(y_1^k; x^B, q^{-\zeta} t_x) \\
& + \mathrm{i}^{-2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_1^n y_1 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} d_1^n y_2\, (\phi_1)'_m(y_2^A, q^{-\zeta} t_2) \\
& \qquad \overset{y_2}{\circledast} V(y_2^j) \overset{y_2}{\circledast} (K_1)'_{m^+}(y_2^k; y_1^B, q^{-\zeta} t_1) \overset{y_1}{\circledast} V(y_1^l) \\
& \qquad \overset{y_1}{\circledast} (K_1)'_{m^+}(y_1^r; x^C, q^{-\zeta} t_x) + \ldots, \qquad (83)
\end{aligned}
$$

$$
\begin{aligned}
(\psi_2)_{m^-}(x^i) = {} & (\phi_2)_m(x^A, q^{\zeta} t_x) \\
& + \mathrm{i}^{-1} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_2^n y_1\, (K_2)_{m^-}(x^A, q^{\zeta} t_x; y_1^j) \overset{y_1}{\circledast} V(y_1^k) \\
& \qquad \overset{y_1}{\circledast} (\phi_2)'_m(y_1^B, q^{\zeta} t_1) \\
& + \mathrm{i}^{-2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_2^n y_1 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} d_2^n y_2\, (K_2)_{m^-}(x^A, q^{\zeta} t_x; y_1^j) \\
& \qquad \overset{y_1}{\circledast} V(y_1^k) \overset{y_1}{\circledast} (K_2)_{m^-}(y_1^B, q^{\zeta} t_1; y_2^l) \overset{y_2}{\circledast} V(y_2^r) \\
& \qquad \overset{y_2}{\circledast} (\phi_2)_m(y_2^C, q^{\zeta} t_2) + \ldots, \qquad (84)
\end{aligned}
$$

and

$$
\begin{aligned}
(\psi_1^*)_{m^+}(x^i) = \; & (\phi_1^*)_m(x^i) \\
+ \; \mathrm{i}^{-1} & \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_1^n y_1 \, (K_1^*)_{m^+}(x^i, y_1^j) \overset{y_1}{\circledast} V(\kappa y_1^A, t_1) \overset{y_1}{\circledast} (\phi_1^*)_m(\kappa y_1^B, t_1) \\
+ \; \mathrm{i}^{-2} & \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_1^n y_1 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} d_1^n y_2 \, (K_1^*)_{m^+}(x^i, y_1^j) \\
& \quad \overset{y_1}{\circledast} V(\kappa y_1^A, t_1) \overset{y_1}{\circledast} (K_1^*)_{m^+}(\kappa y_1^B, t_1; y_2^k) \overset{y_2}{\circledast} V(\kappa y_2^C, t_2) \\
& \quad \overset{y_2}{\circledast} (\phi_1^*)_m(\kappa y_2^D, t_2) + \ldots ,
\end{aligned} \tag{85}
$$

$$
\begin{aligned}
(\psi_2^*)'_m(x^i) = \; & (\phi_2^*)'_m(x^i) \\
+ \; \mathrm{i}^{-1} & \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_2^n y_1 \, (\phi_2^*)'_m(\kappa^{-1} y_1^A, t_1) \overset{y_1}{\circledast} V(\kappa^{-1} y_1^B, t_1) \\
& \quad \overset{y_1}{\circledast} (K_2^*)'_{m^-}(y_1^k, x^i) \\
+ \; \mathrm{i}^{-2} & \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_2^n y_1 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} d_2^n y_2 \, (\phi_2^*)'_m(\kappa^{-1} y_2^A, t_2) \\
& \quad \overset{y_2}{\circledast} V(\kappa y_2^B, t_2) \overset{y_2}{\circledast} (K_2^*)'_{m^-}(y_2^j; \kappa^{-1} y_1^C, t_1) \overset{y_1}{\circledast} V(\kappa^{-1} y_1^B, t_1) \\
& \quad \overset{y_1}{\circledast} (K_2^*)'_{m^-}(y_1^k, x^i) + \ldots .
\end{aligned} \tag{86}
$$

Let us mention that these expressions describe wave functions that emerge from free-particle states in the remote past or future.

A short look at the identities in (52) and (53) should make it obvious that the solutions in (83)-(86) correspond to the expansions

$$
\begin{aligned}
(G_1)'_{m^+}(z^i, x^j) = \; & (K_1)'_{m^+}(z^i; x^A, q^{-\zeta} t_x) \\
+ \; \mathrm{i}^{-1} & \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_1^n y_1 \, (K_1)'_{m^+}(z^i; y_1^B, q^{-\zeta} t_1) \overset{y_1}{\circledast} V(y_1^k) \\
& \quad \overset{y_1}{\circledast} (K_1)'_{m^+}(y_1^l; x^A, q^{-\zeta} t_x) \\
+ \; \mathrm{i}^{-2} & \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_1^n y_1 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} d_1^n y_2 \, (K_1)'_{m^+}(z^i; y_2^C, q^{-\zeta} t_2) \\
& \quad \overset{y_2}{\circledast} V(y_2^k) \overset{y_2}{\circledast} (K_1)'_{m^+}(y_2^l; y_1^B, q^{-\zeta} t_1) \overset{y_1}{\circledast} V(y_1^r) \\
& \quad \overset{y_1}{\circledast} (K_1)'_{m^+}(y_1^s; x^A, q^{-\zeta} t_x) + \ldots ,
\end{aligned} \tag{87}
$$

$$
(G_2)_{m^-}(x^i, z^j) = \; (K_2)_{m^-}(x^A, q^{\zeta} t_x; z^j)
$$

$$
\begin{aligned}
&+ \mathrm{i}^{-1} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_2^n y_1 \, (K_2)_{m^-}(x^A, q^\zeta t_x; y_1^k) \overset{y_1}{\circledast} V(y_1^l) \\
&\qquad \overset{y_1}{\circledast} (K_2)_{m^-}(y_1^B, q^\zeta t_1; z^j) \\
&+ \mathrm{i}^{-2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_2^n y_1 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} d_2^n y_2 \, (K_2)_{m^-}(x^A, q^\zeta t_x; y_1^k) \\
&\qquad \overset{y_1}{\circledast} V(y_1^l) \overset{y_1}{\circledast} (K_2)_{m^-}(y_1^B, q^\zeta t_1; y_2^r) \overset{y_2}{\circledast} V(y_2^s) \\
&\qquad \overset{y_2}{\circledast} (K_2)_{m^-}(y_2^C, q^\zeta t_2; z^j) + \ldots ,
\end{aligned} \tag{88}
$$

and

$$
\begin{aligned}
(G_1^*)_{m^+}(x^i, z^j) = {} & (K_1^*)_{m^+}(x^i, z^j) \\
&+ \mathrm{i}^{-1} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_1^n y_1 \, (K_1^*)_{m^+}(x^i, y_1^j) \overset{y_1}{\circledast} V(\kappa y_1^A, t_1) \\
&\qquad \overset{y_1}{\circledast} (K_1^*)_{m^+}(\kappa y_1^B, t_1; z^j) \\
&+ \mathrm{i}^{-2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_1^n y_1 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} d_1^n y_2 \, (K_1^*)_{m^+}(x^i, y_1^k) \\
&\qquad \overset{y_1}{\circledast} V(\kappa y_1^A, t_1) \overset{y_1}{\circledast} (K_1^*)_{m^+}(\kappa y_1^B, t_1; y_2^l) \overset{y_2}{\circledast} V(\kappa y_2^C, t_2) \\
&\qquad \overset{y_2}{\circledast} (K_1^*)_{m^+}(\kappa y_2^D, t_2; z^j) + \ldots ,
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
(G_2^*)'_{m^-}(z^i, x^j) = {} & (K_2^*)'_{m^-}(z^i, x^j) \\
&+ \mathrm{i}^{-1} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_2^n y_1 \, (K_2^*)'_m(z^i; \kappa^{-1} y_1^A, t_1) \overset{y_1}{\circledast} V(\kappa^{-1} y_1^B, t_1) \\
&\qquad \overset{y_1}{\circledast} (K_2^*)'_{m^-}(y_1^k, x^j) \\
&+ \mathrm{i}^{-2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} d_2^n y_1 \int_{-\infty}^{+\infty} dt_2 \int_{-\infty}^{+\infty} d_2^n y_2 \, (K_2^*)'_{m^-}(z^i; \kappa^{-1} y_2^A, t_2) \\
&\qquad \overset{y_2}{\circledast} V(\kappa^{-1} y_2^B, t_2) \overset{y_2}{\circledast} (K_2^*)'_{m^-}(y_2^k; \kappa^{-1} y_1^C, t_1) \overset{y_1}{\circledast} V(\kappa^{-1} y_1^D, t_1) \\
&\qquad \overset{y_1}{\circledast} (K_2^*)'_{m^-}(y_1^l, x^j) + \ldots .
\end{aligned} \tag{90}
$$

One should also notice that for each geometry we considered either retarded or advanced Green's functions. This correspondence is a consequence of the choice of boundary conditions in (44) and (45). If we change these boundary conditions, the free-particle Green's functions in the above expressions have to be substituted by their counterparts that propagate wave

functions oppositely in time.

In what follows it is necessary to know the conjugation properties of the Green's functions introduced in (52) and (53) ($i = 1, 2$):

$$\begin{aligned}\overline{(G_i)_{m^\pm}(x^k, z^l)} &= (\tilde{G}_i)'_{m^\pm}(z^l, x^k), \\ \overline{(G_i^*)_{m^\pm}(x^k, z^l)} &= (\tilde{G}_i^*)'_{m^\pm}(z^l, x^k),\end{aligned} \tag{91}$$

$$\begin{aligned}\overline{(G_i)'_{m^\pm}(z^k, x^l)} &= (\tilde{G}_i)'_{m^\pm}(x^l, z^k), \\ \overline{(G_i^*)'_{m^\pm}(z^k, x^l)} &= (\tilde{G}_i^*)'_{m^\pm}(x^l, z^k).\end{aligned} \tag{92}$$

To understand these relationship one first has to realize that the Green's functions with a tilde obey relations in which the free-particle propagators are substituted by those with a tilde. With this observation and the results of (36) and (37) at hand the above conjugation properties follow from conjugating the identities in (87)-(90).

3.2 S-matrices and transition probabilities

In this subsection we would like to write down probability amplitudes for finding a certain free-particle state after the scattering process took place. As we know, these amplitudes establish the S-matrix. In our formalism the S-matrices for the different geometries should become

$$\begin{aligned}(S_2)_-(\phi, \psi) &= \lim_{t \to -\infty} \big\langle (\phi_2^*)_m(x^A, -t_x), (\psi_2)_{m^-}(x^B, t_x) \big\rangle_{2,x} \\ &= \lim_{t_x \to -\infty} \lim_{t_y \to \infty} \int_{-\infty}^{+\infty} d_2^n x \int_{-\infty}^{+\infty} d_2^n y \, \overline{(\phi_2^*)_m(x^A, -t_x)} \\ &\qquad \overset{x}{\circledast} (G_2)_{m^-}(x^B, t_x; y^C, t_y) \overset{y}{\circledast} (\phi_2)_m(y^D, q^\zeta t_y),\end{aligned} \tag{93}$$

$$\begin{aligned}(S_1^*)_+(\phi, \psi) &= \lim_{t \to +\infty} \big\langle (\phi_1)_m(x^A, -t_x), (\psi_1^*)_{m^+}(x^B, t_x) \big\rangle_{1,x} \\ &= \lim_{t_x \to +\infty} \lim_{t_y \to -\infty} \int_{-\infty}^{+\infty} d_1^n x \int_{-\infty}^{+\infty} d_1^n y \, \overline{(\phi_1)_m(x^A, -t_x)} \\ &\qquad \overset{x}{\circledast} (G_1^*)_{m^+}(x^B, t_x; y^C, t_y) \overset{y}{\circledast} (\phi_1^*)_m(\kappa y^D, t_y),\end{aligned} \tag{94}$$

and

$$(S_1)'_+(\phi, \psi) = \lim_{t_x \to +\infty} \big\langle (\psi_1)'_{m^+}(x^A, t_x), (\phi_1^*)'_m(x^B, -t_x) \big\rangle'_{1,x}$$

$$= \lim_{t_x \to +\infty} \lim_{t_y \to -\infty} \int_{-\infty}^{+\infty} d_1^n x \int_{-\infty}^{+\infty} d_1^n y \, (\phi_1)'_m(y^C, q^{-\zeta} t_y) \overset{y}{\circledast} (G_1)'_{m^+}(y^D, t_y; x^A, t_x) \overset{x}{\circledast} \overline{(\phi_1^*)'_m(x^B, -t_x)}, \quad (95)$$

$$(S_2^*)'_-(\phi, \psi) = \lim_{t \to -\infty} \big\langle (\psi_2^*)'_{m^-}(x^A, t_x), (\phi_2)'_m(x^B, -t_x) \big\rangle'_{2,x}$$
$$= \lim_{t_x \to -\infty} \lim_{t_y \to +\infty} \int_{-\infty}^{+\infty} d_2^n x \int_{-\infty}^{+\infty} d_2^n y \, (\phi_2^*)'_m(\kappa^{-1} y^C, t_y) \overset{y}{\circledast} (G_2^*)'_{m^-}(y^D, t_y; x^A, t_x) \overset{x}{\circledast} \overline{(\phi_2)'_m(x^B, -t_x)}. \quad (96)$$

Notice that our S-matrix elements were formulated by means of the sesquilinear forms introduced in part I of this article. Due to the relations in (52) and (53) the S-matrix elements can be expressed by Green's functions. In this manner, our S-matrix elements become dependent on two free-particle wave functions, one for the incoming particle and another one for the outcoming particle. It is important to realize that the wave function for the incoming particle refers to a geometry being different from that for the outcoming particle. However, wave functions of different geometries can move into different directions of time. Finally, one should notice that we have assigned a minus sign to the time argument of the wave function of the outgoing particle. This was done to guarantee that the free-particle wave functions involved in one and the same S-matrix element describe particles moving into the same direction in time

Next, we would like to consider the conjugation properties of S-matrix elements. In this respect, we have

$$\overline{(S_2)_-(\phi, \psi)} = (\tilde{S}_2)'_-(\phi, \psi),$$
$$\overline{(S_1)'_+(\phi, \psi)} = (\tilde{S}_1)'_+(\phi, \psi), \quad (97)$$

$$\overline{(S_1^*)_+(\phi, \psi)} = (\tilde{S}_1^*)'_+(\phi, \psi),$$
$$\overline{(S_2^*)'_-(\phi, \psi)} = (\tilde{S}_2^*)_-(\phi, \psi), \quad (98)$$

where

$$(\tilde{S}_1)_+(\phi, \psi) = \lim_{t_x \to +\infty} \lim_{t_y \to -\infty} \int_{-\infty}^{+\infty} d_1^n x \int_{-\infty}^{+\infty} d_1^n y \, \overline{(\phi_1^*)_m(x^A, -t_x)} \overset{x}{\circledast} (\tilde{G}_1)_{m^+}(x^B, t_x; y^C, t_y) \overset{y}{\circledast} (\phi_1)_m(y^D, q^{-\zeta} t_y), \quad (99)$$

$$(\tilde{S}_2^*)_-(\phi,\psi) = \lim_{t_x\to-\infty}\lim_{t_y\to+\infty}\int_{-\infty}^{+\infty} d_2^n x \int_{-\infty}^{+\infty} d_2^n y\, \overline{(\phi_2)_m(x^A,-t_x)} \overset{x}{\circledast} (\tilde{G}_2^*)_{m^-}(x^B,t_x;y^C,t_y) \overset{y}{\circledast} (\phi_2^*)_m(\kappa^{-1}y^D,t_y), \quad (100)$$

and

$$(\tilde{S}_2)'_-(\phi,\psi) = \lim_{t_x\to-\infty}\lim_{t_y\to+\infty}\int_{-\infty}^{+\infty} d_2^n x \int_{-\infty}^{+\infty} d_2^n y\, (\phi_2)'_m(y^A,q^\zeta t_y) \overset{y}{\circledast} (\tilde{G}_2)'_{m^-}(y^B,t_y;x^C,t_x) \overset{x}{\circledast} \overline{(\phi_2^*)'_m(x^D,-t_x)}, \quad (101)$$

$$(\tilde{S}_1^*)'_+(\phi,\psi) = \lim_{t_x\to+\infty}\lim_{t_y\to-\infty}\int_{-\infty}^{+\infty} d_1^n x \int_{-\infty}^{+\infty} d_1^n y\, (\phi_1^*)'_m(\kappa y^A,t_y) \overset{y}{\circledast} (\tilde{G}_1^*)'_{m^+}(y^B,t_y;x^C,t_x) \overset{x}{\circledast} \overline{(\phi_1)'_m(x^D,-t_x)}. \quad (102)$$

The relations in (97) and (98) can be proved in the following manner:

$$\begin{aligned}\overline{(S_2)_-(\phi,\psi)} &= \lim_{t_x\to-\infty}\overline{\left\langle(\phi_2^*)_m(x^A,-t_x),(\psi_2)_m(x^B,t_x)\right\rangle_{i,x}} \\ &= \lim_{t_x\to-\infty}\lim_{t_y\to+\infty}\int_{-\infty}^{+\infty} d_2^n x \int_{-\infty}^{+\infty} d_2^n y\, \overline{(\phi_i)_m(y^D,q^\zeta t_y)} \\ &\qquad \overset{y}{\circledast} \overline{(G_2)_{m^-}(x^B,t_x;y^C,t_y)} \overset{x}{\circledast} (\phi_2^*)_m(x^A,-t_x) \\ &= \lim_{t_x\to-\infty}\lim_{t_y\to+\infty}\int_{-\infty}^{+\infty} d_2^n x \int_{-\infty}^{+\infty} d_2^n y\, (\phi_2)_m(y^D,q^\zeta t_y) \\ &\qquad \overset{x}{\circledast} (\tilde{G}_2)'_{m^-}(y^C,t_y;x^B,t_x) \overset{x}{\circledast} \overline{(\phi_2^*)'_m(x^A,-t_x)} \\ &= (\tilde{S}_2)'_-(\phi,\psi). \end{aligned} \quad (103)$$

The first and the second step in the above calculation make use of the identities in (93). The third step applies the results of (91) together with the identifications (see part II of the article)

$$\overline{(\phi_i)_m(x^k)} = (\phi_i)'_m(x^k), \qquad \overline{(\phi_i^*)_m(x^k)} = (\phi_i^*)'_m(x^k). \quad (104)$$

Now, we are in a position to introduce transition probabilities. It seems to be reasonable to define these quantities by

$$\begin{aligned}(\omega_2)_-(\phi,\psi) &\equiv \overline{(S_2)_-(\phi,\psi)}\cdot(S_2)_-(\phi,\psi), \\ (\omega_1^*)_+(\phi,\psi) &\equiv \overline{(S_1^*)_+(\phi,\psi)}\cdot(S_1^*)_+(\phi,\psi),\end{aligned} \quad (105)$$

$$
\begin{aligned}
(\omega_1)'_+(\phi,\psi) &\equiv (S_1)'_+(\phi),\psi)\cdot\overline{(S_1)'_+(\phi,\psi)},\\
(\omega_2^*)'_-(\phi,\psi) &\equiv (S_2^*)'_-(\phi,\psi)\cdot\overline{(S_2^*)'_-(\phi,\psi)}.
\end{aligned} \tag{106}
$$

From their very definition it follows that the transition probabilities are real.

Alternatively, we can introduce transition probabilities by

$$
\begin{aligned}
(\tilde{\omega}_1)_-(\phi,\psi) &\equiv \overline{(\tilde{S}_1)_+(\phi,\psi)}\cdot(\tilde{S}_1)_+(\phi,\psi),\\
(\tilde{\omega}_2^*)_-(\phi,\psi) &\equiv \overline{(\tilde{S}_2^*)_-(\phi,\psi)}\cdot(\tilde{S}_2^*)_-(\phi,\psi),
\end{aligned} \tag{107}
$$

$$
\begin{aligned}
(\tilde{\omega}_2)'_-(\phi,\psi) &\equiv (\tilde{S}_2)'_-(\phi,\psi)\cdot\overline{(\tilde{S}_2)'_-(\phi,\psi)},\\
(\tilde{\omega}_1^*)'_+(\phi,\psi) &\equiv (\tilde{S}_1^*)'_+(\phi,\psi)\cdot\overline{(\tilde{S}_1^*)'_+(\phi,\psi)}.
\end{aligned} \tag{108}
$$

They are linked to those in (105) and (106) via

$$
\begin{aligned}
(\omega_2)_-(\phi,\psi) &= (\tilde{\omega}_2)'_-(\phi,\psi),\\
(\omega_1^*)_+(\phi,\psi) &= (\tilde{\omega}_1^*)'_+(\phi,\psi),
\end{aligned} \tag{109}
$$

$$
\begin{aligned}
(\omega_1)'_+(\phi,\psi) &= (\tilde{\omega}_1)_+(\phi,\psi),\\
(\omega_2^*)'_-(\phi,\psi) &= (\tilde{\omega}_2^*)_-(\phi,\psi).
\end{aligned} \tag{110}
$$

These identifications follow from the relations in (97) and (98).

Now, we come to the question what becomes of unitarity of the S-matrix in a q-deformed setting. To answer this question it is convenient to determine the S-matrix in an orthonormal basis of q-deformed momentum eigenfunctions. Towards this end we take the expansions of free-particle wave functions in terms of plane waves and insert them into the defining expressions of the S-matrices [cf. (93)-(96)]. This way, we obtain, for example,

$$
\begin{aligned}
&(S_2)_-(\phi,\psi) =\\
&\quad= \lim_{t_x\to-\infty}\lim_{t_y\to+\infty}\int_{-\infty}^{+\infty} d_2^n x\int_{-\infty}^{+\infty} d_2^n y\,\overline{(\phi_2^*)_m(x^A,-t_x)}\\
&\qquad\overset{x}{\circledast}(G_2)_{m^-}(x^B,t_x;y^C,t_y)\overset{y}{\circledast}(\phi_2)_m(y^D,q^\zeta t_y)\\
&\quad= \lim_{t_x\to-\infty}\lim_{t_y\to+\infty}\int_{-\infty}^{+\infty} d_2^n x\int_{-\infty}^{+\infty} d_2^n y\,(\mathrm{vol}_2)^{1/2}\Big(\int_{-\infty}^{+\infty} d_2^n p\,(c_2^*)'_{\kappa p}\\
&\qquad\overset{p}{\circledast}(\bar{u}_{R,\bar{L}})_{p,m}(x^A,-t_x)\Big)\overset{x}{\circledast}(G_2)_{m^-}(x^B,t_x;y^C,t_y)\\
&\qquad\overset{y}{\circledast}\frac{\kappa^{-n}}{(\mathrm{vol}_2)^{1/2}}\int_{-\infty}^{+\infty} d_2^n p'\,(\bar{u}_{R,\bar{L}})_{\ominus_R p',m}(y^D,q^\zeta t_y)\overset{y|p'}{\odot}_L(c_2)_{\kappa^{-1}p'}
\end{aligned}
$$

$$= \int_{-\infty}^{+\infty} d_2^n p \int_{-\infty}^{+\infty} d_2^n p' \, (c_2^*)'_{\kappa p} \overset{p}{\circledast} [(S_2)_-]_{p(\kappa p')} \overset{p'}{\circledast} (c_2)_{p'}, \tag{111}$$

where we introduced as elements of the S-matrix in a momentum basis

$$\begin{aligned}[(S_2)_-]_{pp'} = \lim_{t_x \to -\infty} \lim_{t_y \to +\infty} \int_{-\infty}^{+\infty} d_2^n x \int_{-\infty}^{+\infty} d_2^n y \, (\bar{u}_{R,\bar{L}})_{p,m}(x^A, -t_x) \\ \overset{x}{\circledast} (G_2)_{m^-}(x^B, t_x; y^C, t_y) \\ \overset{y|p'}{\odot}_L (\bar{u}_{R,\bar{L}})_{\ominus_R p',m}(y^D, \kappa^{-2} q^{-\zeta} t_y). \end{aligned} \tag{112}$$

With the same reasonings we find for the other geometries that

$$\begin{aligned}(S_1^*)_+(\phi, \psi) = \\ = \int_{-\infty}^{+\infty} d_1^n p \int_{-\infty}^{+\infty} d_1^n p' \, (c_1)'_p \overset{p}{\circledast} [(S_1^*)_+]_{(\kappa^{-1}p)p'} \overset{p'}{\circledast} (c_1^*)_{\kappa^{-1}p'}, \end{aligned} \tag{113}$$

$$\begin{aligned}(S_1)'_+(\phi, \psi) = \\ = \int_{-\infty}^{+\infty} d_1^n p \int_{-\infty}^{+\infty} d_1^n p' \, (c_1)'_p \overset{p}{\circledast} [(S_1)'_+]_{(\kappa^{-1}p)p'} \overset{p'}{\circledast} (c_1^*)_{\kappa^{-1}p'}, \end{aligned} \tag{114}$$

$$\begin{aligned}(S_2^*)'_-(\phi, \psi) = \\ = \int_{-\infty}^{+\infty} d_2^n p \int_{-\infty}^{+\infty} d_2^n p' \, (c_2^*)'_{\kappa p} \overset{p}{\circledast} [(S_2^*)'_-]_{p(\kappa p')} \overset{p'}{\circledast} (c_2)_{p'}, \end{aligned} \tag{115}$$

where

$$\begin{aligned}[(S_1^*)_+]_{pp'} = \lim_{t_x \to +\infty} \lim_{t_y \to -\infty} \int_{-\infty}^{+\infty} d_1^n x \int_{-\infty}^{+\infty} d_1^n y \, (u_{\bar{R},L})_{\ominus_L p,m}(x^A, -\kappa^2 t_x) \\ \overset{p|x}{\odot}_{\bar{L}} (G_1^*)_{m^\pm}(x^B, t_x; y^C, t_y) \\ \overset{y}{\circledast} (u_{\bar{R},L})_{p',m}(y^D, t_y), \end{aligned} \tag{116}$$

$$\begin{aligned}[(S_1)'_+]_{pp'} = \lim_{t_x \to +\infty} \lim_{t_y \to -\infty} \int_{-\infty}^{+\infty} d_1^n x \int_{-\infty}^{+\infty} d_1^n y \, (u_{\bar{R},L})_{\ominus_L p,m}(y^A, \kappa^2 q^{-\zeta} t_y) \\ \overset{p|y}{\odot}_R (G_1)'_{m^+}(y^B, t_y; x^C, t_x) \\ \overset{x}{\circledast} (u_{\bar{R},L})_{p',m}(x^D, -t_x), \end{aligned} \tag{117}$$

$$\begin{aligned}
[(S_2^*)'_-]_{pp'} = \lim_{t_x\to-\infty}\lim_{t_y\to+\infty}\int_{-\infty}^{+\infty} d_2^n x \int_{-\infty}^{+\infty} d_2^n y\, (\bar{u}_{R,\bar{L}})_{p,m}(y^A, t_y) \\
\overset{y}{\circledast} (G_2^*)'_{m^-}(y^B, t_y; x^C, t_x) \\
\overset{x|p}{\odot}_{\bar{R}} (\bar{u}_{R,\bar{L}})_{\ominus_R p',m}(y^B, -\kappa^{-2}t_y).
\end{aligned} \tag{118}$$

As next step we calculate products of S-matrices in a momentum basis. Concretely, we are interested in products like the following one:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d_2^n p''\, [(S_2)_-]_{p(\kappa p'')} \overset{p''}{\circledast} [(\tilde{S}_2)_+]_{p''(\kappa p')} = \\
&= \lim_{t\to-\infty}\lim_{t'\to+\infty}\int_{-\infty}^{+\infty} d_2^n p'' \int_{-\infty}^{+\infty} d_2^n x_1 \int_{-\infty}^{+\infty} d_2^n y_1\, (\bar{u}_{R,\bar{L}})_{p,m}(x_1^A, -t) \\
&\qquad \overset{x_1}{\circledast} (G_2)_{m^-}(x_1^B, t; y_1^C, t') \overset{y_1|p''}{\odot}_L (\bar{u}_{R,\bar{L}})_{\ominus_R(\kappa p''),m}(y_1^D, \kappa^{-2}q^{\zeta}t') \\
&\qquad \overset{p''}{\circledast} \int_{-\infty}^{+\infty} d_2^n x_2 \int_{-\infty}^{+\infty} d_2^n y_2\, (\bar{u}_{R,\bar{L}})_{p'',m}(x_2^E, -t') \\
&\qquad \overset{x_2}{\circledast} (\tilde{G}_2)_{m^+}(x_2^F, t'; y_2^G, t) \overset{y_2|p'}{\odot}_L (\bar{u}_{R,\bar{L}})_{\ominus_R(\kappa p'),m}(y_2^H, \kappa^{-2}q^{\zeta}t) \\
&= \lim_{t\to-\infty}\lim_{t'\to+\infty}\int_{-\infty}^{+\infty} d_2^n x_1 \int_{-\infty}^{+\infty} d_2^n y_1\, \frac{1}{\mathrm{vol}_2}(\bar{u}_{R,\bar{L}})_{p,m}(x_1^A, -t) \\
&\qquad \overset{x_1}{\circledast} (G_2)_{m^-}(x_1^B, t; y_1^C, t') \overset{y_1}{\circledast} \int_{-\infty}^{+\infty} d_2^n x_2\, \delta_2^n((\ominus_{\bar{L}}(\kappa y_1^D) \oplus_{\bar{L}} x_2^E) \\
&\qquad \overset{x_2}{\circledast} \int_{-\infty}^{+\infty} d_2^n y_2\, (\tilde{G}_2)_{m^+}(x_2^F, t'; y_2^G, t) \overset{y_2|p'}{\odot}_L (\bar{u}_{R,\bar{L}})_{\ominus_R(\kappa p'),m}(y_2^H, \kappa^{-2}q^{\zeta}t) \\
&= \lim_{t\to-\infty}\lim_{t'\to+\infty}\int_{-\infty}^{+\infty} d_2^n x_1 \int_{-\infty}^{+\infty} d_2^n y_1 \int_{-\infty}^{+\infty} d_2^n y_2\, (\bar{u}_{R,\bar{L}})_{p,m}(x_1^A, -t) \\
&\qquad \overset{x_1}{\circledast} (G_2)_{m^-}(x_1^B, t; y_1^C, t') \overset{y_1}{\circledast} (\tilde{G}_2)_{m^+}(x_2^D, t'; y_2^E, t) \\
&\qquad \overset{y_2|p'}{\odot}_L (\bar{u}_{R,\bar{L}})_{\ominus_R(\kappa p'),m}(y_2^F, \kappa^{-2}q^{\zeta}t) \\
&= \frac{\kappa^{-n}}{\mathrm{vol}_2}\int_{-\infty}^{+\infty} d_2^n x_1 \int_{-\infty}^{+\infty} d_2^n y_2\, (\bar{u}_{R,\bar{L}})_{p,m}(x_1^A, -t) \\
&\qquad \overset{x_1}{\circledast} \delta_2^n((\ominus_{\bar{L}}\, x_1^B) \oplus_{\bar{L}} (\kappa^{-1}y_2^C)) \overset{y_2|p'}{\odot}_L (\bar{u}_{R,\bar{L}})_{\ominus_R(\kappa p'),m}(y_2^D, \kappa^{-2}q^{\zeta}t) \\
&= \int_{-\infty}^{+\infty} d_2^n x_1\, (\bar{u}_{R,\bar{L}})_{p,m}(x_1^A, -t) \overset{x_1|p'}{\odot}_L (\bar{u}_{R,\bar{L}})_{\ominus_R(\kappa p'),m}(x_1^B, \kappa^{-2}q^{\zeta}t) \\
&= (\mathrm{vol}_2)^{-1}\delta_2^n(p^A \oplus_R (\ominus_R\, \kappa p'^B)).
\end{aligned} \tag{119}$$

For the first step we plug in the expressions of the S-matrices in a momentum basis. Then we identify the completeness relation of plane waves, i.e.

$$\begin{aligned}\int_{-\infty}^{+\infty} d_2^n p \, (\bar{u}_{R,\bar{L}})_{\ominus_R p,m}(y^B, q^\zeta t) \overset{y|p}{\odot}_{\bar{R}} (\bar{u}_{R,\bar{L}})_{p,m}(x^A, -t) = \\ = (\mathrm{vol}_2)^{-1} \delta_2^n((\ominus_{\bar{L}} \, y^B) \oplus_{\bar{L}} x^A). \end{aligned} \tag{120}$$

In this manner, we get a q-deformed delta function. Exploiting its characteristic property

$$\int_{-\infty}^{+\infty} d_2^n x \, \delta_2^n((\ominus_{\bar{L}} \, y^A) \oplus_{\bar{L}} x^B) \overset{x}{\circledast} f(x^C) = \mathrm{vol}_2 \, f(\kappa^{-1} y^A), \tag{121}$$

one integration vanishes. Then the fourth step uses the identities in (74) and leads to another q-deformed delta function. For this reason a further integral disappears. The last step is nothing other than orthonormality of plane waves. For these reasonings to become more clear we additionally added their diagrammatic version in Figs. 1 and 2 (for an introduction into this subject we refer the reader to Ref. [18]).

After repeating these steps for the other geometries we should end up with the following list of relations:

$$\begin{aligned}&\int_{-\infty}^{+\infty} d_2^n p'' \, [(S_2)_-]_{p(\kappa p'')} \overset{p''}{\circledast} [(\tilde{S}_2)_+]_{p''(\kappa p')} = \\ &\quad = \int_{-\infty}^{+\infty} d_2^n p'' \, [(\tilde{S}_2)_+]_{p(\kappa p'')} \overset{p''}{\circledast} [(S_2)_-]_{p''(\kappa p')} \\ &\quad = (\mathrm{vol}_2)^{-1} \delta_2^n(p^A \oplus_R (\ominus_R \, \kappa p'^B)), \end{aligned} \tag{122}$$

$$\begin{aligned}&\int_{-\infty}^{+\infty} d_1^n p'' \, [(S_1^*)_+]_{(\kappa^{-1}p)p''} \overset{p''}{\circledast} [(\tilde{S}_1^*)_-]_{(\kappa^{-1}p'')p'} = \\ &\quad = \int_{-\infty}^{+\infty} d_1^n p'' \, [(\tilde{S}_1^*)_-]_{(\kappa^{-1}p)p''} \overset{p''}{\circledast} [(S_1^*)_+]_{(\kappa^{-1}p'')p'} \\ &\quad = (\mathrm{vol}_1)^{-1} \delta_1^n((\ominus_L \, \kappa^{-1} p^A) \oplus_L p'^B), \end{aligned} \tag{123}$$

and

$$\begin{aligned}&\int_{-\infty}^{+\infty} d_1^n p'' \, [(S_1)'_+]_{(\kappa^{-1}p)p''} \overset{p''}{\circledast} [(\tilde{S}_1)'_-]_{(\kappa^{-1}p'')p'} = \\ &\quad = \int_{-\infty}^{+\infty} d_1^n p'' \, [(\tilde{S}_1)'_-]_{(\kappa^{-1}p)p''} \overset{p''}{\circledast} [(S_1)'_+]_{(\kappa^{-1}p'')p'} \end{aligned}$$

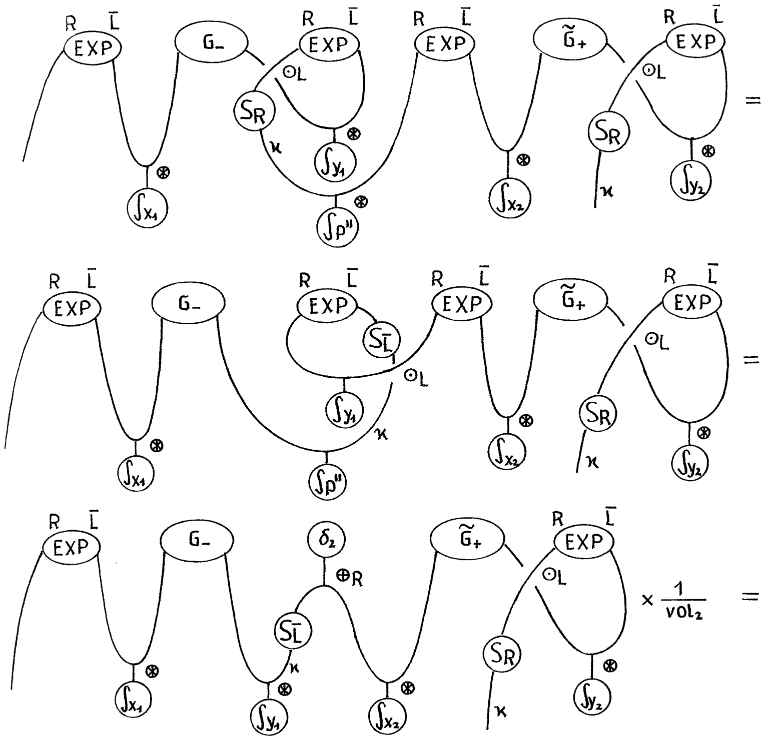

Figure 1: First part of a diagrammatic proof corresponding to (119).

$$= (\mathrm{vol}_1)^{-1}\delta_1^n((\ominus_L \kappa^{-1}p^A) \oplus_L p'^B), \tag{124}$$

$$\begin{aligned}\int_{-\infty}^{+\infty} d_2^m p'' \, [(S_2^*)'_-]_{p(\kappa p'')} \overset{p''}{\circledast} [(\tilde{S}_2^*)'_+]_{p''(\kappa p')} &= \\ = \int_{-\infty}^{+\infty} d_2^m p'' \, [(\tilde{S}_2^*)'_+]_{p(\kappa p'')} \overset{p''}{\circledast} [(S_2^*)'_-]_{p''(\kappa p')} \\ = (\mathrm{vol}_2)^{-1}\delta_2^n(p^A \oplus_R (\ominus_R \kappa p'^B)). \end{aligned} \tag{125}$$

The above relations tell us that in a momentum basis the S-matrices are invertible. For each S-matrix the corresponding inverse can be read off from the results in (122)-(125).

Interestingly, the inverse of a given S-matrix can be identified with the Hermitian conjugate of another S-matrix. In this sense, we have

$$\overline{[(S_2)_-]_{pp'}} = [(\tilde{S}_2)'_-]_{p'p}, \qquad \overline{[(S_1^*)_+]_{pp'}} = [(\tilde{S}_1^*)'_+]_{p'p},$$

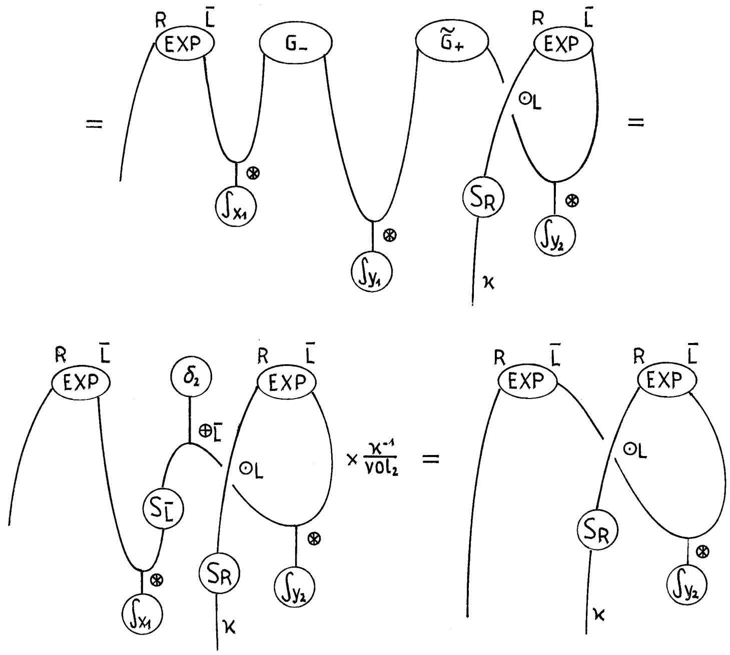

Figure 2: Second part of a diagrammatic proof corresponding to (119).

$$\overline{[(S_1)'_+]_{pp'}} = [(\tilde{S}_1)_+]_{p'p}, \qquad \overline{[(S^*_2)'_-]_{pp'}} = [(\tilde{S}^*_2)_-]_{p'p}. \tag{126}$$

These equalities can be readily checked if we insert the expressions for the S-matrix elements and take into account the conjugation properties of Green's functions and plane waves.

Using the conjugation properties in (126) the relations in (122)-(125) can be rewritten as

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d^n_2 p''\, [(S_2)_-]_{p(\kappa p'')} \overset{p''}{\circledast} \overline{[(S_2)'_+]_{p'p''}} = \\
&\quad = \int_{-\infty}^{+\infty} d^n_2 p''\, \overline{[(S_2)'_+]_{(\kappa p'')p}} \overset{p''}{\circledast} [(S_2)_-]_{p''p'} \\
&\quad = (\mathrm{vol}_2)^{-1} \delta^n_2 (p^A \oplus_R (\ominus_R\, p'^B)),
\end{aligned} \tag{127}$$

$$\int_{-\infty}^{+\infty} d^n_1 p''\, [(S^*_1)_+]_{pp''} \overset{p''}{\circledast} \overline{[(S^*_1)'_-]_{p'(\kappa^{-1}p'')}} =$$

$$= \int_{-\infty}^{+\infty} d_1^n p'' \, \overline{[(S_1^*)'_-]_{p''p}} \overset{p''}{\circledast} [(S_1^*)_+]_{(\kappa^{-1}p'')p'}$$
$$= (\mathrm{vol}_1)^{-1} \delta_1^n((\ominus_L p^A) \oplus_L p'^B), \tag{128}$$

and

$$\int_{-\infty}^{+\infty} d_1^n p'' \, [(S_1)'_+]_{pp''} \overset{p''}{\circledast} \overline{[(S_1)_-]_{p'(\kappa^{-1}p'')}} =$$
$$= \int_{-\infty}^{+\infty} d_1^n p'' \, \overline{[(S_1)_-]_{p''p}} \overset{p''}{\circledast} [(S_1)'_+]_{(\kappa^{-1}p'')p'}$$
$$= (\mathrm{vol}_1)^{-1} \delta_1^n((\ominus_L p^A) \oplus_L p'^B), \tag{129}$$

$$\int_{-\infty}^{+\infty} d_2^n p'' \, [(S_2^*)'_-]_{p(\kappa p'')} \overset{p''}{\circledast} \overline{[(S_2^*)_+]_{p'p''}} =$$
$$= \int_{-\infty}^{+\infty} d_2^n p'' \, \overline{[(S_2^*)_+]_{(\kappa p'')p}} \overset{p''}{\circledast} [(S_2^*)'_-]_{p''p'}$$
$$= (\mathrm{vol}_2)^{-1} \delta_2^n(p^A \oplus_R (\ominus_R p'^B)). \tag{130}$$

In this manner, we arrived at identities that express unitarity of S-matrices in the setting of q-deformation.

Finally, we would like to remind the reader of the fact that in this section we dealt with certain geometries, only. The expressions corresponding to the other geometries are again obtained by applying the substitutions in (9) to the results of this section.

4 The interaction picture

In this section we would like to show that our formalism allows to consider the process of scattering from a point of view provided by the interaction picture. Let us recall that the interaction picture is nothing other than a reformulation of the Schrödinger picture. Particularly, it becomes very useful in describing scattering processes.

In a scattering process we usually start from free-particle states, i.e. their time-evolution is determined by a free-particle Hamiltonian H_0. Such a free-particle state can be expanded in terms of momentum eigenstates and the corresponding expansion coefficients describe to which extent the different momentum eigenstates are populated. Notice that in our formalism momentum eigenstates are represented by q-deformed plane waves.

If the particle interacts with a potential V its wave function can again

be expanded in terms of momentum eigenfunctions, since q-deformed plane waves establish a complete and orthonormal set of functions. Under the influence of the potential V this expansion can change in time. For this reason the expansion coefficients now show an additional time-dependence which contains information about the probability for a particle to be forced into a certain momentum state.

The time dependence of the plane waves, however, can be neglected, since it does not carry any information about the interaction. In this manner, the wave functions describing the scattering process in the interaction picture should become

$$\begin{aligned}(\Psi_1)'_m(x^i) &= \frac{\kappa^n}{(\mathrm{vol}_1)^{1/2}} \int_{-\infty}^{+\infty} d_1^n p\, (C_1)'_{\kappa p}(t) \overset{p|x}{\odot}_R (u_{\bar{R},L})_{\ominus_L p,m}(x^A, t=0) \\ &= \frac{\kappa^n}{(\mathrm{vol}_1)^{1/2}} \big\langle (C_1)'_{\kappa p}(t), (\bar{u}_{\bar{R},L})_{\ominus_{\bar{R}} p,m}(x^A, t=0) \big\rangle'_{1,p}, \end{aligned} \tag{131}$$

$$\begin{aligned}(\Psi_2)_m(x^i) &= \frac{\kappa^{-n}}{(\mathrm{vol}_2)^{1/2}} \int_{-\infty}^{+\infty} d_2^n p\, (\bar{u}_{R,\bar{L}})_{\ominus_R p,m}(x^A, t=0)) \overset{x|p}{\odot}_L (C_2)_{\kappa^{-1} p}(t) \\ &= \frac{\kappa^{-n}}{(\mathrm{vol}_2)^{1/2}} \big\langle (u_{R,\bar{L}})_{\ominus_{\bar{L}} p,m}(x^A, t=0), (C_2)_{\kappa^{-1} p}(t) \big\rangle_{2,p}, \end{aligned} \tag{132}$$

and

$$\begin{aligned}(\Psi_1^*)_m(x^i) &= (\mathrm{vol}_1)^{1/2} \int_{-\infty}^{+\infty} d_1^n p\, (u_{\bar{R},L})_{p,m}(x^A, t=0) \overset{p}{\circledast} (C_1^*)_{\kappa^{-1} p}(t) \\ &= (\mathrm{vol}_1)^{1/2} \big\langle (\bar{u}_{\bar{R},L})_{p,m}(x^A, t=0), (C_1^*)_{\kappa^{-1} p}(t) \big\rangle_{1,p}, \end{aligned} \tag{133}$$

$$\begin{aligned}(\Psi_2^*)'_m(x^i) &= (\mathrm{vol}_2)^{1/2} \int_{-\infty}^{+\infty} d_2^n p\, (C_2^*)'_{\kappa p}(t) \overset{p}{\circledast} (\bar{u}_{R,\bar{L}})_{p,m}(x^A, t=0) \\ &= (\mathrm{vol}_2)^{1/2} \big\langle (C_2^*)'_{\kappa p}(t), (u_{R,\bar{L}})_{p,m}(x^A, t=0) \big\rangle'_{2,p}. \end{aligned} \tag{134}$$

Notice that throughout this section we take the convention that wave functions and expansion coefficients referring to the interaction picture are written in capital letters.

The relationship between wave functions of the interaction picture and those of the Schrödinger picture can be expressed as

$$\begin{aligned}(\Psi_1)'_m(x^i) &= \exp(\mathrm{i}q^{-\zeta} t H_0) \overset{x}{\triangleright} (\psi_1)'_m(x^i), \\ (\Psi_1^*)_m(x^i) &= \exp(\mathrm{i}t H_0) \overset{x}{\triangleright} (\psi_1^*)_m(x^i), \end{aligned} \tag{135}$$

and

$$
\begin{aligned}
(\Psi_2)_m(x^i) &= (\psi_2)_m(x^i) \overset{x}{\triangleleft} \exp(-\mathrm{i}q^{\zeta}H_0 t), \\
(\Psi_2^*)'_m(x^i) &= (\psi_2^*)'_m(x^i) \overset{x}{\triangleleft} \exp(-\mathrm{i}H_0 t). \qquad (136)
\end{aligned}
$$

Let us note that in the above relations the operators on the right-hand side remove the time-dependence resulting from the free-particle Hamiltonian H_0.

With the very same reasonings as in the undeformed case (see for example Ref. [53]) it follows from the identities in (135) and (136) that we have

$$
\begin{aligned}
\mathrm{i}\partial_0 \overset{t}{\triangleright} (\Psi_1)'_m(x^i) &= (V_1)'_I \overset{x}{\triangleright} (\Psi_1)'_m(x^i), \\
\mathrm{i}\partial_0 \overset{t}{\triangleright} (\Psi_1^*)_m(x^i) &= (V_1^*)_I \overset{x}{\triangleright} (\Psi_1^*)_m(x^i), \qquad (137)
\end{aligned}
$$

and

$$
\begin{aligned}
(\Psi_2)_m(x^i) \overset{t}{\triangleleft} (\mathrm{i}\hat{\partial}_0) &= (\Psi_2)_m(x^i) \overset{x}{\triangleleft} (V_2)_I, \\
(\Psi_2^*)'_m(x^i) \overset{t}{\triangleleft} (\mathrm{i}\hat{\partial}_0) &= (\Psi_2^*)'_m(x^i) \overset{x}{\triangleleft} (V_2^*)_I, \qquad (138)
\end{aligned}
$$

where

$$
\begin{aligned}
(V_1)'_I &= \exp(\mathrm{i}tq^{-\zeta}H_0)V\exp(-\mathrm{i}tq^{-\zeta}H_0), \\
(V_2)_I &= \exp(-\mathrm{i}tq^{\zeta}H_0)V\exp(\mathrm{i}tq^{\zeta}H_0), \qquad (139)
\end{aligned}
$$

and

$$
\begin{aligned}
(V_1^*)_I &= \exp(\mathrm{i}tH_0)V\exp(-\mathrm{i}tH_0), \\
(V_2^*)'_I &= \exp(-\mathrm{i}tH_0)V\exp(\mathrm{i}tH_0). \qquad (140)
\end{aligned}
$$

To find solutions to the differential equations in (137) and (138) we introduce time-evolution operators for the wave functions of the interaction picture, i.e.

$$
\begin{aligned}
(\Psi_1)'_m(x^A,t) &= (U_1)'_I(t,\tilde{t}) \overset{x}{\triangleright} (\Psi_1)'_m(x^A,\tilde{t}), \\
(\Psi_1^*)_m(x^A,t) &= (U_1^*)_I(t,\tilde{t}) \overset{x}{\triangleright} (\Psi_1^*)_m(x^A,\tilde{t}), \qquad (141)
\end{aligned}
$$

and

$$
\begin{aligned}
(\Psi_2)_m(x^A,t) &= (\Psi_2)_m(x^A,\tilde{t}) \overset{x}{\triangleleft} (U_2)_I(t,\tilde{t}),\\
(\Psi_2^*)'_m(x^A,t) &= (\Psi_2^*)'_m(x^A,\tilde{t}) \overset{x}{\triangleleft} (U_2^*)'_I(t,\tilde{t}).
\end{aligned} \tag{142}
$$

Inserting these expressions into the differential equations in (137) and (138) yields differential equations for the time evolution operators in the interaction picture:

$$
\begin{aligned}
\mathrm{i}\partial_0 \overset{t}{\triangleright} (U_1)'_I(t,\tilde{t}) &= (V_1)'_I(t)\,(U_1)'_I(t,\tilde{t}),\\
\mathrm{i}\partial_0 \overset{t}{\triangleright} (U_1^*)_I(t,\tilde{t}) &= (V_1^*)_I(t)\,(U_1^*)_I(t,\tilde{t}),
\end{aligned} \tag{143}
$$

and

$$
\begin{aligned}
(U_2)_I(t,\tilde{t}) \overset{t}{\triangleleft} (\mathrm{i}\hat{\partial}_0) &= (U_2)_I(t,\tilde{t})\,(V_2)_I(t),\\
(U_2^*)'_I(t,\tilde{t}) \overset{t}{\triangleleft} (\mathrm{i}\hat{\partial}_0) &= (U_2^*)'_I(t,\tilde{t})\,(V_2^*)_I(t).
\end{aligned} \tag{144}
$$

If we require

$$
(U_1)'_I(t,t) = (U_1^*)_I(t,t) = (U_2)_I(t,t) = (U_2^*)'_I(t,t) = 1, \tag{145}
$$

the differential equations in (143) and (144) are equivalent to the integral equations

$$
\begin{aligned}
(U_1)'_I(t,\tilde{t}) &= 1 - \mathrm{i}\int_{\tilde{t}}^{t} dt'\,(V_1)'_I(t')\,(U_1)'_I(t',\tilde{t}),\\
(U_1^*)_I(t,\tilde{t}) &= 1 - \mathrm{i}\int_{\tilde{t}}^{t} dt'\,(V_1^*)_I(t')\,(U_1^*)_I(t',\tilde{t}),
\end{aligned} \tag{146}
$$

and

$$
\begin{aligned}
(U_2)_I(t,\tilde{t}) &= 1 + \mathrm{i}\int_{t'}^{t} dt'\,(U_2)_I(t',\tilde{t})\,(V_2)_I(t'),\\
(U_2^*)'_I(t,\tilde{t}) &= 1 + \mathrm{i}\int_{t'}^{t} dt'\,(U_2^*)'_I(t',\tilde{t})\,(V_2^*)'_I(t'),
\end{aligned}
$$

By iteration we find as formal solutions

$$
(U_1)'_I(t,\tilde{t}) = 1 + \sum_{n=1}^{\infty} \mathrm{i}^{-n} \int_{\tilde{t}}^{t} dt_1 \int_{\tilde{t}}^{t_1} dt_2 \ldots \int_{\tilde{t}}^{t_{n-1}} dt_n\,(V_1)'_I(t_1)\ldots(V_1)'_I(t_n),
$$

$$(U_1^*)_I(t,\tilde{t}) = 1 + \sum_{n=1}^{\infty} \mathrm{i}^{-n} \int_{\tilde{t}}^{t} dt_1 \int_{\tilde{t}}^{t_1} dt_2 \ldots \int_{\tilde{t}}^{t_{n-1}} dt_n\, (V_1^*)_I(t_1) \ldots (V_1^*)_I(t_n), \tag{147}$$

and

$$\begin{aligned}(U_2)_I(t,\tilde{t}) &= 1 + \sum_{n=1}^{\infty} \mathrm{i}^{n} \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \ldots \int_{t'}^{t_{n-1}} dt_n\, (V_2)_I(t_n) \ldots (V_2)_I(t_1),\\ (U_2^*)'_I(t,\tilde{t}) &= 1 + \sum_{n=1}^{\infty} \mathrm{i}^{n} \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \ldots \int_{t'}^{t_{n-1}} dt_n\, (V_2^*)_I(t_n) \ldots (V_2^*)_I(t_1).\end{aligned} \tag{148}$$

It remains to write down formulae that extract the coefficients from the expansions in (131)-(134). Recalling the results in Ref. [49] this task can be achieved by the formulae

$$\begin{aligned}(C_1)'_p(t) &= (\mathrm{vol}_1)^{1/2} \int_{-\infty}^{+\infty} d_1^n x\, (\Psi_1)'_m(x^i) \overset{x}{\circledast} (u_{\bar{R},L})_{p,m}(x^A, t=0)\\ &= (\mathrm{vol}_1)^{1/2} \big\langle (\Psi_1)'_m(x^i), (\bar{u}_{\bar{R},L})_{p,m}(x^A, t=0) \big\rangle'_{1,x},\end{aligned} \tag{149}$$

$$\begin{aligned}(C_2)_p(t) &= (\mathrm{vol}_2)^{1/2} \int_{-\infty}^{+\infty} d_2^n x\, (\bar{u}_{R,\bar{L}})_{p,m}(x^A, t=0) \overset{x}{\circledast} (\Psi_2)_m(x^i)\\ &= (\mathrm{vol}_2)^{1/2} \big\langle (u_{R,\bar{L}})_{p,m}(x^A, t=0), (\Psi_2)_m(x^i) \big\rangle_{2,x},\end{aligned} \tag{150}$$

and

$$\begin{aligned}(C_1^*)_p(t) &= (\mathrm{vol}_1)^{-1/2} \int_{-\infty}^{+\infty} d_1^n x\, (u_{\bar{R},L})_{\ominus_{\bar{R}} p,m}(x^A, t=0) \overset{p|x}{\odot}_{\bar{L}} (\Psi_1^*)_m(x^i)\\ &= (\mathrm{vol}_1)^{-1/2} \big\langle (\bar{u}_{\bar{R},L})_{\ominus_L p,m}(x^A, t=0), (\Psi_1^*)_m(x^i) \big\rangle_{1,x},\end{aligned} \tag{151}$$

$$\begin{aligned}(C_2^*)'_p(t) &= (\mathrm{vol}_2)^{-1/2} \int_{-\infty}^{+\infty} d_2^n x\, (\Psi_2^*)'_m(x^i) \overset{x|p}{\odot}_{\bar{R}} (\bar{u}_{\bar{R},L})_{\ominus_{\bar{L}} p,m}(x^A, t=0)\\ &= (\mathrm{vol}_2)^{-1/2} \big\langle (\Psi_2^*)'_m(x^i), (u_{\bar{R},L})_{\ominus_R p,m}(x^A, t=0) \big\rangle_{2,x}.\end{aligned} \tag{152}$$

These equalities are a direct consequence of the completeness relations for momentum eigenfunctions. In this sense, they can be viewed as a kind of inverse Fourier transformation.

As already mentioned the expansion coefficients in (149)-(152) are nothing other than transition amplitudes which contain information about find-

ing a system in a certain momentum eigenstate. In what follows we would like to adapt these ideas in a way suitable for describing scattering processes.

Towards this end let us first recall that in the interaction picture free-particle wave functions are time-independent. Thus, they are linked to the free-particle wave functions of the Schrödinger picture via the relations

$$\begin{aligned}(\Phi_1)'_m(x^A) &= (\phi_1)'_m(x^A, t_x = 0),\\ (\Phi_2)_m(x^A) &= (\phi_2)_m(x^A, t_x = 0),\end{aligned} \tag{153}$$

$$\begin{aligned}(\Phi^*_1)_m(x^A) &= (\phi^*_1)_m(x^A, t_x = 0),\\ (\Phi^*_2)'_m(x^A) &= (\phi^*_2)'_m(x^A, t_x = 0).\end{aligned} \tag{154}$$

In scattering processes one requires for the particles to be free in the remote past or remote future. For wave functions of the interaction picture these conditions read

$$\begin{aligned}\lim_{t\to+\infty}(\Psi_2)_{m^-}(x^A, t) &= (\Phi_2)_m(x^A),\\ \lim_{t\to-\infty}(\Psi_1)'_{m^+}(x^A, t) &= (\Phi_1)'_m(x^A),\end{aligned} \tag{155}$$

$$\begin{aligned}\lim_{t\to-\infty}(\Psi^*_1)_{m^+}(x^A, t) &= (\Phi^*_1)_m(x^A),\\ \lim_{t\to+\infty}(\Psi^*_2)'_{m^-}(x^A, t) &= (\Phi^*_2)'_m(x^A).\end{aligned} \tag{156}$$

The amplitudes that determine the probability for detecting a certain free particle state are given by

$$\begin{aligned}(S_2)_-(\Phi, \Psi) &= \lim_{t\to-\infty}\big\langle(\Phi^*_2)_m(x^A), (\Psi_2)_{m^-}(x^B, t)\big\rangle_{2,x}\\ &= \big\langle(\Phi^*_2)_m(x^A), (\Phi_2)_m(x^B) \overset{x}{\triangleleft} (U_2)_I(-\infty, +\infty)\big\rangle_{2,x},\end{aligned} \tag{157}$$

$$\begin{aligned}(S^*_1)_+(\Phi, \Psi) &= \lim_{t\to+\infty}\big\langle(\Phi_1)_m(x^A), (\Psi^*_1)_{m^+}(x^B, t)\big\rangle_{1,x}\\ &= \big\langle(\Phi_1)_m(x^A), (U^*_1)_I(+\infty, -\infty) \overset{x}{\triangleright} (\Phi^*_1)_m(x^B, t)\big\rangle_{1,x},\end{aligned} \tag{158}$$

and

$$\begin{aligned}(S_1)'_+(\Phi, \Psi) &= \lim_{t\to+\infty}\big\langle(\Psi_1)'_{m^+}(x^A, t), (\Phi^*_1)'_m(x^B)\big\rangle'_{1,x}\\ &= \big\langle(U_1)'_I(+\infty, -\infty) \overset{x}{\triangleright} (\Phi_1)'_m(x^A), (\Phi^*_1)'_m(x^B)\big\rangle'_{1,x},\end{aligned} \tag{159}$$

$$
\begin{aligned}
(S_2^*)'_-(\Phi,\Psi) &= \lim_{t\to-\infty} \big\langle (\Psi_2^*)'_{m^-}(x^A,t), (\Phi_2)'_m(x^B)\big\rangle'_{2,x} \\
&= \big\langle (\Phi_2^*)'_m(x^A) \overset{x}{\triangleleft} (U_2^*)'_I(-\infty,+\infty), (\Phi_2)'_m(x^B)\big\rangle'_{2,x}. \qquad (160)
\end{aligned}
$$

The second equality in each of the above equations follows from the relations in (141) and (142) together with the boundary conditions in (155) and (156).

5 Conclusion

Let us make some comments on the results of our examinations about a q-analog of non-relativistic Schrödinger theory.

In part I of the present paper we presented a mathematical and physical framework that can lead to theories with the attractive feature that space is discretized but time is not. Based on these ideas we worked out the basics of a non-relativistic Schrödinger theory on q-deformed quantum spaces as the braided line and the three-dimensional q-deformed Euclidean space. This was done in part II and III of our article.

Contrary to many other approaches for introducing a lattice-like structure in physics our theory does not suffer from the absence of important space-time symmetries, such as rotational or translational symmetry. For this reason, it can be developed along the same line of reasonings as its undeformed counterpart. In view of this observations our q-deformed version of Schrödinger theory seems to be of the same value as the classical non-relativistic Schrödinger theory, which we regain from our results when the deformation parameter q tends to 1.

However, q-deformation of physical theories gives rise to some more structure. For example, we saw that q-deformation requires to distinguish different geometries, which become identical in the undeformed case. To get a complete description of physical phenomena one cannot restrict attention to one such geometry. This circumstance makes things more difficult, but it can lead to new physical phenomena.

On the other hand a special quality of our approach is that it generalizes and extends a well established theory in a way that the relationship to the undeformed limit is rather clear. So to speak, the classical theory can be seen as an approximation or simplification of a more detailed description. In this manner, our q-deformed version of non-relativistic Schrödinger theory can be a useful step in examining and understanding the implications of q-deformation on quantum mechanics, since the treatment of more realistic space-time structures like the q-deformed Minkowski space [35–39] is very awkward (for other deformations of space-time see also Refs. [54–60]).

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